MTH 235: HOMEWORK 2

Scoring. As per Professor Bailey's email, each problem is now worth 3 points, where 3 points are awarded if the problem is entirely correct, including a sufficient amount of justification; 2 points are awarded if the student shows a general understanding of the main ideas of the problem, but their argument is flawed or not sufficiently justified; 1 point otherwise if the problem is at all attempted; 0 points are only given if the student does not attempt the problem. Also, if you award 1 or 2 points for a problem be sure to let the student know *why*.

Problem (Section 1.4; Problem 4)**.** *For each list of polynomials in* $\mathcal{P}_3(\mathbb{R})$*, determine whether the first polynomial can be expressed as a linear combination of the other two.*

- *a)*$x^3 - 3x + 5$*;* $x^3 + 2x^2 - x + 1$*;* $x^3 + 3x^2 - 1$.
- *c)*$-2x^3 - 11x^2 + 3x + 2$*;* $x^3 - 2x^2 + 3x - 1$*;* $2x^3 + x^2 + 3x - 2$.
- *e)*$x^3 - 8x^2 + 4x$*;* $x^3 - 2x^2 + 3x - 1$*;* $x^3 - 2x + 3$.

Proof. a)

$$\begin{aligned} x^3 - 3x + 5 &= \alpha(x^3 + 2x^2 - x + 1) + \beta(x^3 + 3x^2 - 1) \\ &= (\alpha + \beta)x^3 + (2\alpha + 3\beta)x^2 - \alpha x + (\alpha - \beta) \end{aligned}$$

so,

$$\alpha = 3, \beta = -2$$

and the system is consistent because the other two equations reduce to tautologies, i. e. $0 = 0$.

c)

$$\begin{aligned} -2x^3 - 11x^2 + 3x + 2 &= \alpha(x^3 - 2x^2 + 3x - 1) + \beta(2x^3 + x^2 + 3x - 2) \\ &= (\alpha + 2\beta)x^3 + (-2\alpha + \beta)x^2 + (3\alpha + 3\beta)x + (-\alpha - 2\alpha) \end{aligned}$$

so,

$$\alpha = 4, \beta = -3$$

and the system is consistent because the other two equations reduce to tautologies.

e)

$$\begin{aligned} x^3 - 8x^2 + 4x &= \alpha(x^3 - 2x^2 + 3x - 1) + \beta(x^3 - 2x + 3) \\ &= (\alpha + \beta)x^3 - 2\alpha x^2 + (3\alpha - 2\beta)x + (-\alpha + 3\beta). \end{aligned}$$

In fact, there are no such α, β since the system is inconsistent, that is, at least one of the equations reduces to a false statement. ☐

Problem (Section 1.4; Problem 11)**.** *Prove that* $span(\{x\}) = \{ax : a \in F\}$ *for any vector* x *in a vector space. Interpret this result geometrically in* $\mathbb{R}^3$.

Proof. By definition $span(\{x\})$ is the set of all linear combinations of the elements of the set $\{x\}$, that is, $span(\{x\}) = \{ax : a \in F\}$. In $\mathbb{R}^3$, $span(\{x\})$ for $x \in \mathbb{R}^3$ is the line through the origin containing the vector x. ☐

Problem (Section 1.4; Problem 16). *Let V be a vector space and S a subset of V with the property that whenever $v_1, \dots, v_n \in S$ and $a_1v_1 + \cdots + a_nv_n = 0$, then $a_1 = \cdots = a_n = 0$. Prove that every vector in the span of S can be* uniquely *written as a linear combination of vectors of S.*

Proof. Let $S = \{v_1, \dots, v_n\}$. Then for $x \in span(\{x\})$, assume there exist $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

and

$$x = \beta_1 v_1 + \cdots + \beta_n v_n.$$

Then we can equate the above expressions, and, by axioms $VS4$ and $VS8$, we find

$$(\alpha_1 - \beta_1)v_1 + \cdots + (\alpha_n - \beta_n)v_n = 0.$$

By hypothesis, we conclude that $\alpha_i = \beta_i$ for all $i = 1, \dots, n$, that is, every vector in the span of S can be uniquely written as a linear combination of the vectors of S. □

Problem (Section 1.5; Problem 12). *Prove Theorem* 1.6 *and its corollary.*

- Theorem 1.6:*Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.*
- Corollary: *Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.*

Proof. Theorem 1.6: Assume that $S_1 = \{v_1, \dots, v_m\}$ and that $S_2 = \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ where $m \leq n$. Then S_1 linearly dependent means we can write one of the vectors $v_1, \dots, v_m$ as a nontrivial linear combination of the other vectors. For simplicity, assume that

$$v_1 = \alpha_2 v_2 + \cdots + \alpha_m v_m,$$

where at least one of the α_i, $i = 2, \dots, m$ is nonzero. Then by $VS3$, we can write

$$\begin{aligned} v_1 &= \alpha_2 v_2 + \cdots + \alpha_m v_m + 0 \\ &= \alpha_2 v_2 + \cdots + \alpha_m v_m + 0 * v_{m+1} + \cdots + 0 * v_n. \end{aligned}$$

Since at least one of the α_i, $i = 2, \dots, m$ is nonzero, this is a nontrivial linear combination of the remaining vectors in S_2. Thus, S_2 is linearly dependent.

Corollary: The easiest way to prove the corollary is to simply note that it is the logical contrapositive of the above theorem. That is, "if S_1 is linearly dependent, then S_2 is linearly dependent," is logically equivalent to "if S_2 is not linearly dependent, then S_1 is not linearly dependent." Of course, since a set is linearly independent if it is not linearly dependent, this statement is exactly the corollary.

A second proof is based on the idea of the proof of the above theorem. Assume that S_1 and S_2 are as above. If S_2 is linearly independent, then $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Now, assume that $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$. Then as above, we can write

$$\begin{aligned} 0 &= \alpha_1 v_1 + \cdots + \alpha_m v_m \\ &= \alpha_1 v_1 + \cdots + \alpha_m v_m + 0 \\ &= \alpha_1 v_1 + \cdots + \alpha_m v_m + 0 * v_{m+1} + \cdots + 0 * v_n. \end{aligned}$$

Hence, we conclude $\alpha_1 = \cdots = \alpha_m = 0$ by the linear independence assumption on S_2, which proves S_1 is linear independent. □

Problem (Section 1.5; Problem 17). *Let M be a square upper triangular matrix with nonzero diagonal entries. Prove that the columns of M are linearly independent.*

Proof. Let M be an $n \times n$ upper triangular matrix. Let M_i denote the i^{th} column of M; then M_i is the column vector whose entries below the i^{th} row are all 0. Then assume there exist $\beta_1, \ldots, \beta_n$ such that

$$0 = \beta_1 M_1 + \cdots + \beta_n M_n,$$

which gives the following system of equations

$$\begin{aligned} 0 &= \beta_1 \alpha_{11} + \cdots + \beta_n \alpha_{1n}; \\ 0 &= \beta_2 \alpha_{22} + \cdots + \beta_n \alpha_{2n}; \\ &\vdots \\ 0 &= \beta_n \alpha_{nn}. \end{aligned}$$

Starting from the last equation, it is easy to see that $\beta_n = \beta_{n-1} = \cdots = \beta_1 = 0$. Hence, the columns of an upper triangular matrix are linearly independent. □