MTH 235: HOMEWORK 1

Problem (Section 1.1; Problem 7)**.** *Prove that the diagonals of a parallelogram bisect each other.*

Proof. Let O denote the origin in $\mathbb{R}^2$; let $OPQR$ be the parallelogram determined by the vector u, with starting point O and terminal point P, and the vector v, with starting point O and terminal point R. Assume the following coordinates:

$$P = (a_1, b_1);$$
$$Q = (a_2, b_2);$$
$$R = (a_3, b_3).$$

From this, note that $P = (a_1, b_1) = u$, $R = (a_3, b_3) = v$, so $Q = (a_2, b_2) = u + v = (a_1 + a_3, b_1 + b_3)$; that is, we have the following equalities:

$$a_2 = a_1 + a_3 \tag{1}$$

and

$$b_2 = b_1 + b_3. \tag{2}$$

By Problem 6 of this section, the midpoint of the line segment OQ is given by

$$(\frac{a_2}{2}, \frac{b_2}{2}).$$

From (1) and (2), this is equivalent to the point given by

$$(\frac{a_1 + a_3}{2}, \frac{b_1 + b_3}{2}),$$

but this is, again by Problem 6, exactly the midpoint of the line segment PR. □

Scoring.
- Give 2 points for any logically sound argument;
- Give only 1 point for an argument that contains logical flaws but that shows a basic understanding of the idea involved;
- 0 points for no argument or an argument that shows no understanding of the problem.

Problem (Section 1.2; Problem 9)**.** *Prove Corollaries* 1/2 *of Theorem* 1.1 *and Theorem* 1.2*c).*

Corollary (1)**.** *The vector* 0 *such that* $x + 0 = x$ *for each* $x \in V$ *is unique.*

Corollary (2)**.** *The vector* y *such that* $x + y = 0$ *is unique.*

Theorem (1.2c)**.** $a \cdot 0 = 0$ *for each* $a \in F$ *where* 0 *is the zero vector.*

Proof. ·Corollary 1: Assume that $\tilde{0}$ is another vector such that $x + \tilde{0} = x$. Then we have that

$$\begin{aligned} 0 + x &= x + 0 = x \\ &= x + \tilde{0} = \tilde{0} + x, \end{aligned}$$

where the first and fourth equalities follow from $VS1$, the second from $VS3$ and the third by assumption. By Theorem 1.1, we find that $0 = \tilde{0}$, so that 0 is unique.

·Corollary 2: Given $x \in V$, let $y, \tilde{y} \in V$ be two vectors such that $x + y = 0$ and $x + \tilde{y} = 0$. Then we have that

$$\begin{aligned} y + x &= x + y = 0 \\ &= x + \tilde{y} = \tilde{y} + x, \end{aligned}$$

where the first and fourth equalities follows from $VS1$ and the second and third by our assumptions. By Theorem 1.1, $y = \tilde{y}$, so that y is unique.

·Theorem 1.2c: Let a be any element of the field F and 0 the zero vector of the vector space V over F. Then we have the following

$$\begin{aligned} a \cdot 0 + a \cdot 0 &= a(0 + 0) \\ &= a \cdot 0 = a \cdot 0 + 0 = 0 + a \cdot 0 \end{aligned}$$

where the first equality follows from $VS8$, the second and third by $VS3$ and the fourth from $VS1$. The result now follows by applying Theorem 1.1. □

Scoring. For each of the three proofs, give 1 point for a correct argument, .5 points if they show the calculation but don't justify it and 0 points for no argument or a flawed argument, so that there is a total of 3 possible points for this problem.

Problem (Section 1.2; Problem 21)**.** *Let V, W be vector spaces over the field F. Let $Z = \{(v, w) : v \in V, w \in W\}$. Prove that Z is itself a vector space over F with the operations*

- $(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2)$*;*
- $c \cdot (v, w) = (c \cdot v, c \cdot w)$*.*

Proof. Throughout, assume that $(v, w), (v_1, w_1), (v_2, w_2), (v_3, w_3) \in Z$ and $\alpha, \beta \in F$.

·VS1:

$$\begin{aligned} (v_1, w_1) + (v_2, w_2) &= (v_1 + v_2, w_1 + w_2) \\ &= (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1) \end{aligned}$$

where the first and last equalities are the definition of + on Z and the middle equality follows by using $VS1$ on V and W.

·VS2:

$$\begin{aligned} (v_1, w_1) + [(v_2, w_2) + (v_3, w_3)] &= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) = ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \\ &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = [(v_1, w_1) + (v_2, w_2)] + (v_3, w_e) \end{aligned}$$

where the first, second, fourth and fifth equalities are just the definition of + on Z and the third equality follows from $VS2$ on V and W.

·VS3: Let 0_V, 0_W denote the zero vectors in V, W respectively.

$$(v, w) + (0_V, 0_W) = (v + 0_V, w + 0_W) = (v, w)$$

where the first equality is the definition of + on Z and the second follows from $VS3$ on V and W.

·VS4: For $(v, w) \in Z$, we have $(-v, -w) \in Z$ because $-v \in V$ and $-w \in W$ by $VS4$ on V and W. Then

$$(v, w) + (-v, -w) = (v + (-v), w + (-w)) = (0_V, 0_W)$$

where the first equality is the definition of + on Z and the second follows from $VS4$ on V and W.

$\cdot$<u>VS5</u>: By $VS5$ on V and W, there is a $1 \in F$ such that $1 \cdot v = v$ and $1 \cdot w = w$ for each $v \in V$ and each $w \in W$. Then by definition of $\cdot$ on Z we have

$$1 \cdot (v, w) = (1 \cdot v, 1 \cdot w) = (v, w).$$

$\cdot$<u>VS6</u>:

$$\begin{aligned}(\alpha\beta) \cdot (v, w) &= ((\alpha\beta) \cdot v, (\alpha\beta) \cdot w) \\ &= (\alpha \cdot (\beta \cdot v), \alpha \cdot (\beta \cdot w)) = \alpha \cdot (\beta \cdot v, \beta \cdot w) \\ &= \alpha \cdot (\beta \cdot (v, w))\end{aligned}$$

where the first, third and fourth equalities follow from the definition of $\cdot$ on Z and the second equality from $VS6$ on V and W.

$\cdot$<u>VS7</u>:

$$\begin{aligned}\alpha \cdot [(v_1, w_1) + (v_2, w_2)] &= \alpha \cdot (v_1 + v_2, w_1 + w_2) \\ &= (\alpha \cdot (v_1 + v_2), \alpha \cdot (w_1 + w_2)) = (\alpha \cdot v_1 + \alpha \cdot v_2, \alpha \cdot w_1 + \alpha \cdot w_2) \\ &= (\alpha \cdot v_1, \alpha \cdot w_1) + (\alpha \cdot v_2, \alpha \cdot w_2) = \alpha \cdot (v_1, w_1) + \alpha \cdot (v_2, w_2)\end{aligned}$$

where the first and fourth equalities follow from the definition of $+$ on Z, the second and fifth equalities from the definition of $\cdot$ on Z, the third from $VS7$ on V and W.

$\cdot$<u>VS8</u>:

$$\begin{aligned}(\alpha + \beta) \cdot (v, w) &= ((\alpha + \beta) \cdot v, (\alpha + \beta) \cdot w) \\ &= (\alpha \cdot v + \beta \cdot v, \alpha \cdot w + \beta \cdot w) = (\alpha \cdot v, \alpha \cdot w) + (\beta \cdot v, \beta \cdot w) \\ &= \alpha \cdot (v, w) + \beta \cdot (v, w)\end{aligned}$$

where the first and fourth equalities follow from the definition of $\cdot$ on Z, the second equality from $VS8$ on V and W and the third equality from the definition of $+$ on Z. □

Scoring.
- 1 point for checking *all* the conditions for a vector space *and* providing justifications for over half them;
- .5 points for checking *all* the conditions with little to no justification;
- 0 points for no answer or for not checking all the conditions.

Problem (Section 1.3; Problem 12)**.** *An $m \times n$ matrix is called* upper triangular *if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the set of upper triangular matrices form a subspace of $M_{m\times n}(F)$.*

Proof. The zero matrix is upper triangular since all entries, and hence in particular, all the entries lying below the diagonal entries, are zero. Now let A, B be two upper triangular matrices; then for $i > j$,

$$(A + B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0$$

so that the set of upper triangular matrices are closed under addition. Let $\alpha \in F$ and A be an upper triangular matrix; then for $i > j$,

$$(\alpha \cdot A)_{ij} = \alpha \cdot A_{ij} = \alpha \cdot 0 = 0$$

so that the set of upper triangular matrices are closed under scalar multiplication; therefore, we conclude that the set of upper triangular matrices form a subspace of $M_{m\times n}(F)$. □

Scoring.
- 3 points for checking all three conditions to be a subspace;
- 2 points for checking two;

- 1 point for checking one;
- 0 for no argument.

Problem (Section 1.3; Problem 28). *A matrix M is called* skew-symmetric *if $M^t = -M$. Clearly a skew-symmetric matrix must be square. Let F be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from F is a subspace of $M_{n\times n}(F)$. Now assume that F is not of characteristic 2, and let W_2 be the subspace of $M_{n\times n}(F)$ consisting of all symmetric $n \times n$ matrices. Prove that $M_{n\times n}(F) = W_1 \oplus W_2$.*

Proof. The zero matrix 0 is a skew-symmetric matrix. For $A, B \in W_1$

$$\begin{aligned}((A+B)^t)_{ij} &= (A^t + B^t)_{ij} = (A^t)_{ij} + (B^t)_{ij} \\ &= -A_{ij} + -B_{ij} = -(A_{ij} + B_{ij}) \\ &= -(A+B)_{ij}\end{aligned}$$

so that W_1 is closed under addition. For $\alpha \in F$ and $A \in W_1$,

$$\begin{aligned}((\alpha \cdot A)^t)_{ij} &= (\alpha \cdot A^t)_{ij} = \alpha \cdot (A^t)_{ij} \\ &= \alpha \cdot (-A_{ij}) = -\alpha \cdot A_{ij} = -(\alpha \cdot A)_{ij}\end{aligned}$$

so that W_1 is closed under scalar multiplication. Hence W_1 is a subspace of $M_{n\times n}(F)$.

Now, let $A \in W_1 \cap W_2$, so that $(A^t)_{ij} = A_ij$, that is A is symmetric, and $(A^t)_{ij} = -A_{ij}$, that is A is skew-symmetric. From those two equations, we see that

$$A_{ij} = -A_{ij}$$

which implies that

$$0 = A_{ij} + A_{ij} = 2A_{ij}$$

so that $A_{ij} = 0$, that is A must be the zero matrix. It is clear that $W_1 + W_2 \subset V$.

Now, for $A \in M_{n\times n}(F)$, let B_1, B_2 be the two matrices given by

$$B_1 = \frac{1}{2}(A + A^t)$$

and

$$B_2 = \frac{1}{2}(A - A^t).$$

Then

$$\begin{aligned}B_1^t &= [\frac{1}{2}(A + A^t)]^t = \frac{1}{2}(A^t + (A^t)^t) \\ &= \frac{1}{2}(A^t + A) = \frac{1}{2}(A + A^t) = B_1,\end{aligned}$$

and

$$\begin{aligned}B_2^t &= [\frac{1}{2}(A - A^t)]^t = \frac{1}{2}(A^t - (A^t)^t) \\ &= \frac{1}{2}(A^t - A) = -\frac{1}{2}(A - A^t) = -B_2;\end{aligned}$$

that is, B_1 is a symmetric matrix and B_2 is a skew-symmetric matrix. Finally, an easy calculation shows that $B_1 + B_2 = A$. Thus, the space of $n \times n$ matrices is equal to the direct sum of the spaces of symmetric and skew-symmetric matrices. □

Scoring.
- Give 1 point for checking each of the three conditions for W_1 to be a subspace;

- Give .5 point for checking the $W_1 \cap W_2 = \{0\}$;
- Give .5 points for noting that $W_1 + W_2 \subset M_{n\times n}(F)$;
- Give 2 points for a correct argument that any $n \times n$ matrix is the sum of a symmetric and skew-symmetric matrix.

Thus there is a total of 6 possible points on this problem; there are 15 possible points for the entire set.