MTH 235: Linear Algebra

1^{st} Midterm Exam Solutions

October 9, 2009

Part A

1. (10 points) Determine whether the following are subspaces.

a) Let $A \in M_{n\times n}(F)$. The set $S = \{B \in M_{n\times n}(F) \,|\, AB + B = 0\}$.

b) $S = \{f \in P_3(F) \,|\, \text{f(x)=0 or } \deg f(x) = 3\}$.

Solution.

a) This is a subspace. Indeed, let $B, C \in S$ and $a \in F$. In particular, $AB + B = 0$ and $AC + C = 0$.

$$\begin{aligned} A(B+C) + (B+C) &= AB + AC + B + C \\ &= AB + B + AC + C \\ &= (AB + B) + (AC + C) \\ &= 0 + 0 = 0 \end{aligned}$$

so S is closed under addition. Furthermore,

$$\begin{aligned} A(aB) + (aB) &= a(AB) + aB \\ &= a(AB + B) \\ &= a \cdot 0 = 0 \end{aligned}$$

so S is closed under scalar multiplication. Hence S is a subspace.

b) Take $x^3 + x^2, -x^3 \in S$, their sum

$$(x^3 + x^2) + (-x^3) = x^2 \notin S.$$

S is not closed under addition, hence is not a subspace.

2. (10 points) Let u, v, and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

Solution. We would like to show that if

$$a(u+v+w)+b(v+w)+c(w)=0$$

for scalars $a, b, c \in F$, then $a=b=c=0$.

$$\begin{aligned} a(u+v+w)+b(v+w)+c(w)&=0 \\ au+(a+b)v+(a+b+c)w&=0 \end{aligned}$$

since it is given that $\{u, v, w\}$ independent, the only representation of the zero vector is the trivial one so that

$$\begin{aligned} a&=0 \\ a+b&=0 \\ a+b+c&=0 \end{aligned}$$

by back-substitution $a=b=c=0$. So $\{u+v+w, v+w, w\}$ is an independent set. Since $\dim V=3$, any linear independent set with three vectors is a basis.

3. (25 points)

a) Show that the collection $\beta = \{1, 1+x, 1+x+x^2\}$ forms a basis for $P_2(\mathbb{R})$.

b) Show that the transformation $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$$

is a linear transformation.

c) Consider the ordered basis

$$\gamma = \left\{ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of $M_{2\times 2}(\mathbb{R})$. What is $[T]_\beta^\gamma$?

d) Is T onto? Is T one-to-one?

e) State the "rank-nullity" (or dimension) theorem. Verify that T obeys the theorem.

Solution.

a) The standard basis for $P_2(\mathbb{R})$ is $\{1, x, x^2\}$. By number 2, $\{1, 1+x, 1+x+x^2\}$ is also a basis.

b) We would like to show that T preserves addition and scalar multiplication. Let $f, g \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$.

$$\begin{aligned}
T\left((cf+g)(x)\right) &= \begin{pmatrix} (cf+g)'(0) & 2(cf+g)(1) \\ 0 & (cf+g)''(3) \end{pmatrix} \\
&= \begin{pmatrix} (cf'+g')(0) & 2(cf+g)(1) \\ 0 & (cf''+g'')(3) \end{pmatrix} \\
&= \begin{pmatrix} cf'(0)+g'(0) & 2cf(1)+2g(1) \\ 0 & cf''(3)+g''(3) \end{pmatrix} \\
&= \begin{pmatrix} cf'(0) & 2cf(1) \\ 0 & cf''(3) \end{pmatrix} + \begin{pmatrix} g'(0) & 2g(1) \\ 0 & g''(3) \end{pmatrix} \\
&= c\begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix} + \begin{pmatrix} g'(0) & 2g(1) \\ 0 & g''(3) \end{pmatrix} \\
&= cT(f(x)) + T(g(x))
\end{aligned}$$

c) Apply T to the basis β and write the resulting matrix as a linear combination of the basis elements in γ.

$$[T(1)]_\gamma = \left[\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$[T(1+x)]_\gamma = \left[\begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 4 \\ 0 \end{pmatrix}$$

$$\left[T(1+x+x^2)\right]_\gamma = \left[\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 6 \\ 2 \end{pmatrix}$$

Giving the matrix representation:

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

d) It is easily seen that $\{T(1), T(1+x), T(1+x+x^2)\}$ is linearly independent set. Since these vectors generate the range of T, $\text{rank}(T) = 3 < \dim M_{2\times 2}(\mathbb{R}) = 4$, this transformation is **not** onto.

Suppose $f(x) = ax^2 + bx + c$ then $f'(0) = 2a + b$, $2f(1) = a + b + c$ and $f''(3) = 2a$. If $T(f(x)) = 0$ then $2a + b = a + b + c = 2a = 0$. Back-substitution gives $a = b = c = 0$ so f is the 0 polynomial. Hence T is one-to-one.

e) The rank-nullity theorem states that if V is a finite dimensional vector space and T is a linear transformation then

$$\text{rank}(T) + \text{null}(T) = \dim V$$

In part (d), we saw that the rank(T)=3. Since T is one-to-one the kernel of T is the zero subspace, so null(T)=0. The rank-nullity theorem works out since $\dim(P_2(\mathbb{R})) = 3$.

4. (15 points) Circle **T**rue or **F**alse. Any "ambiguous" circles will be marked incorrect, so make sure your answer is clear.

T [**F**] A vector space may have more than one zero vector.

[**T**] **F** The zero vector is a linear combination of any nonempty set of vectors.

T [**F**] If S is a linearly independent set, then each vector of S is a linear combination of other vectors of S.

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T [**F**] The zero vector space has no basis.

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[**T**] **F** If U and W are subspaces of a vector space V, then $U \cap W$ is a subspace of V.

[**T**] **F** If $\text{span}(\{v_1, v_2, \ldots, v_n\}) = \mathbb{R}^n$ then $\{v_1, v_2, \cdots, v_n\}$ is a linearly independent set.

T [**F**] A vector space cannot have more than one basis.

[**T**] **F** If a vector space has a finite basis, then the number of vectors in every basis is the same.

[**T**] **F** If T is linear, then T preserves sums and scalar products.

[**T**] **F** If $T, U : V \to W$ are both linear and agree on a basis for V then $T = U$.

T [**F**] Suppose $T : V \to W$ is a linear transformation. If $T(x) = T(y)$ then $x = y$.

[**T**] **F** If $T : V \to W$ is a linear transformation and $v \in \ker(T)$ then $T(u + v) = T(u)$ for all $u \in V$.