MTH 235 : Linear Algebra

Sample problems - Midterm 1

Note that §1.7 is not included in the practice problems. It will not be tested on the midterm. Other than these questions, make sure you **look at the true/false questions** (#1 of each section listed below). True/False questions will be included on the midterm!

§1.2–1.3 Vector spaces and subspaces.

1. Let V be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Show that $W \subset V$ is a subspace, where
   (a) $W = \{f \mid f(1) = 0\}$
   (b) $W = \{f \mid f(3) = f(1)\}$
   (c) $W = \{f \mid f(-x) = -f(x)\}$
2. Prove that $W_0 = \{(a_1, a_2, \ldots, a_n) \in F^n \mid a_1 + a_2 + \cdots + a_n = 0\}$ is a subspace of F^n, but $W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n \mid a_1 + a_2 + \cdots + a_n = 1\}$ is not.
3. Let $T : V \to W$ be a linear transformation. Prove that $\ker T \subset V$ is a subspace.
4. Fix $A \in M_{n\times n}(\mathbb{R})$. Consider the subset
   $$S = \{B \in M_{n\times n}(\mathbb{R}) \mid AB = BA\}.$$
   Show that S is a subspace.

§1.4–1.6 Linear combinations, independence, bases and dimension.

1. If possible, find a solution of the system of equations:
   $$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 0 \\ -2x_1 + x_2 - x_3 &= 2 \\ 2x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$
2. Show that the vectors $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ generate F^3.
3. The following vectors span $\mathbb{R}^3$:
   $$\begin{aligned} u_1 &= (1, 2, 2) \\ u_2 &= (2, 5, 4) \\ u_3 &= (1, 3, 2) \\ u_4 &= (2, 7, 4) \\ u_5 &= (1, 1, 0) \end{aligned}$$

(a) Pare down the set $\{u_1, u_2, u_3, u_4, u_5\}$ to form a basis, β, for $\mathbb{R}^3$.

(b) Find $(5, 4, 7)$ as a linear combination of the vectors in β.

§2.1–2.2 Linear transformations, their null space, range and matrix representations.

1. Prove that T is a linear transformation, and find bases for the kernel and range of T. Is T one-to-one? Is T onto?

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

2. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. What is $T(8, 11)$?

3. Prove that a linear transformation $T : V \to W$ is one-to-one if and only if $\ker T = \{0\}$.

4. Let V and W be finite-dimensional vectors spaces and $T : V \to W$ be linear.

   (a) Prove that if $\dim V < \dim W$, then T cannot be onto.

   (b) Prove that if $\dim V > \dim W$, then T cannot be one-to-one.

5. Consider the linear transformation $L : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ defined by

   $$L(f(x)) = f'(x) + f(0)$$

   Find the matrix representation, $[L]_\beta^\gamma$ if $\beta = \{2, 1 - x\}$ and $\gamma = \{1, x, x^2\}$.

6. Let $T : \mathbb{R}^3 \to R^3$ be the linear transformation such that

   $$T(x_1, x_2, x_3) = (2x_1 - x_2 - x_3, 2x_2 - x_1 - x_3, 2x_3 - x_1 - x_2)$$

   Find $[T]_\beta$ where β is the standard ordered basis of $\mathbb{R}^3$.