MATH 218

PRACTICE MIDTERM EXAM

Oct 21, 2009

NAME (please print legibly): ______________________________

- **No calculators are allowed on this exam.**
- **A 8 inch by 11 inch notesheet (both sides) may be used for this exam.**
- **Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**
- **Attempt all questions.**

Part-A		
QUESTION	VALUE	SCORE
1	15	
2	15	
3	15	
4	15	
5	10	
6	10	
7	10	
8	10	
TOTAL	100	

Part-A

1. (15 pts)

Find a formula for P_n as a function of n if P_n satisfies the recursion:

$$P_{n+2} = 5P_{n+1} - 6P_n$$

with $P_0 = 10$, $P_1 = 15$.

2. (15 pts)

Demographics Consider a population with 3 age classes called class 0 (people 0-29 years old), class 1 (30-59 years old) and class 2 (60-89 years old) and let a time step be 30 years long. Assume that there are never people outside these age classes so death is assured by age 90. Let P_n^i be the number of people in class i at time step n, for $i = 0, 1, 2$. Let $\hat{P}_n$ be the total population vector at step n given by

$$\hat{P}_n = \begin{bmatrix} P_n^0 \\ P_n^1 \\ P_n^2 \end{bmatrix}.$$

Remember to keep sane that the subscript n in P_n^j refers to the time step you're looking at and the superscript j refers to the age class of individuals you are looking at. Suppose the following data about the population:

The average number of births per person in age class 0 over a time step is 2.1

The average number of births per person in age class 1 over a time step is 1.8

The average number of births per person in age class 2 over a time step is 0.4

The fraction of class 0 that survives and ages to class 1 over a time step is 0.9

The fraction of class 1 that survives and ages to class 2 over a time step is 0.6.

With this picture, the total population vector satisfies:

$$\hat{P}_{n+1} = \mathbb{A}\hat{P}_n$$

where the 3×3 matrix is called the Leslie matrix of this population.

(a) State the Leslie matrix $\mathbb{A}$ for this population explicitly.

(b) Given that the eigenvalues of this matrix are (approximately):

$\lambda_1 = -0.45, \lambda_2 = -0.18, \lambda_3 = 2.72$ with eigenvectors $\hat{v}_1 = \begin{bmatrix} 28 \\ -57 \\ 77 \end{bmatrix}$ $\hat{v}_2 = \begin{bmatrix} 6 \\ -28 \\ 96 \end{bmatrix}$ $\hat{v}_3 = \begin{bmatrix} 95 \\ 31 \\ 7 \end{bmatrix}$

respectively, write down the form for the general solution for P_n as a function of n. (This will be a formula with n and three constants C_1, C_2, C_3.)

(c) In the long run, each component of the population grows at the same geometric rate. What is this long term geometric rate of growth? What is the ratio between classes in the long term?

3. (15 pts)

Consider the system

$$x_{n+1} = x_n - 2y_n$$
$$y_{n+1} = -2x_n + 4y_n$$

If we write $\hat{P}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ then we can write

$$\hat{P}_{n+1} = \mathbb{A}\hat{P}_n$$

where $\mathbb{A} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$.

(a) Given that $\hat{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\hat{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are the two eigenvectors for $\mathbb{A}$, find the corresponding eigenvalues λ_1 and λ_2.

(b) Write down the general form of the solution for $\hat{P}_n$. (This will include two constants C_1, C_2.)

(c) If $x_0 = y_0 = 20$ compute C_1 and C_2. What happens in the long term for the two species in this habitat?

(d) If $x_0 = 30, y_0 = 10$ compute C_1 and C_2. What happens in the long term for the two species in this habitat?

4. (15 pts) Let us consider a 2-allele, non-sex-linked trait and let x_n, y_n, z_n be the fraction of the population in generation n that is homozygous recessive, heterozygous and homozygous dominant respectively. Let $\hat{G}_n = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$ be the genotype vector of the population as usual and let us assume the population progresses according to the Hardy-Weinberg law.

If $\hat{G}_0 = \begin{bmatrix} \frac{1}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{bmatrix}$ what happens to the population as it progresses? In other words state what equilibrium it will approach and also how long it takes to reach this equilibrium.

5. (10 pts) Consider the differential equation

$$\frac{dP}{dt} = P^2 - 8P + 7$$

(a) Find the steady states $S_1 < S_2$ of this DEQ.

(b) Find the sign of $\frac{dP}{dt}$ for $P > S_2$, $S_1 < P < S_2$ and $P < S_1$.

(c) Based on your work in (a) and (b) draw a rough sketch of the behaviour of solutions in the $P - t$ plane. Make sure to draw the steady states. Also state which of these steady states is stable and which is unstable.

6. (10 pts) (a) Write down a continuous analog of the discrete recursion

$$P_{n+1} = 4P_n + 5$$

(b) Write down a discrete analog of the DEQ

$$\frac{dy}{dt} = y^3$$

7. (10 pts) Suppose a population grows according to the exponential growth model

$$\frac{dP}{dt} = 1.4P$$

with $P_0 = 1000$. Write down a formula for the solution. Explain why if $y = ln(P)$ is plotted against t for this population, a straight line will be obtained. State what the y-intercept and slope of this line is in this case.

8. (10 pts) Consider the following DEQ:

$$\frac{dP}{dt} = P^2 t^3$$

Find the steady state solution for P and then integrate the DEQ to find the non-steady state solutions. (Make sure to not forget the integration constant and try to write your final answer in the form $P = h(t)$ for some function $h(t)$.)