UNIVERSITY of ROCHESTER

MTH210: Financial Mathematics

Midterm Exam Solutions

November 3, 2009

Name (please print legibly): ______________________________

University ID Number: ______________________________

Please check the box of your instructor.

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- Do all 6 problems in the space provided and explain your work carefully. The quality of your writeup counts. A correct answer with no work shown will not receive full credit. Please label and circle your final answers.
- You are responsible for checking that this exam has all 6 pages. Please tell us immediately if your exam is missing a page. Missing pages will not contribute to your total score.
- The midterm exam score will be multiplied by three to give a total of 300 points.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Let $A(t)$ be the price of a bond at time t and let $S(t)$ be the price of a stock at time t. Let the present price of the stock be \$37 per share, and suppose we know that, after one time period, its price will be either \$41 or \$33. Suppose that the present price of a bond is \$100 and suppose that after one time period its price will be \$113. Find the price $C(0)$ of a European call option if the strike price K is \$35.

Solution. We have $A(0) = 100$, $A(1) = 113$, $S(0) = 37$, and

$$S(1) = \begin{cases} 41, \\ 33. \end{cases}$$

Here we note that the exercise time is 1 and the payoff $C(1)$ of the call option when the strike price is $K = 39$ is

$$C(1) = \max(S(1) - K, 0) = \begin{cases} 6, & \text{if the stock goes up,} \\ 0, & \text{if the stock goes down.} \end{cases}$$

Now, the replicating portfolio has x shares of stock and y bonds such that

$$\begin{cases} 41x + 113y = 6, & \text{if the stock goes up,} \\ 33x + 113y = 0, & \text{if the stock goes down.} \end{cases}$$

Subtracting the second equation from the first equation, we obtain $8x = 6$. Solving for x, we get that $x = 3/4 = 0.75$. Substituting this into the second equation, we get that $y = -0.219027$. Thus the price of the option is

$$\begin{aligned} C(0) &= S(0)x + A(0)y \\ &= (37)(0.75) + (100)(-0.219027) \\ &= 5.8473. \end{aligned}$$

2. (20 points) The price (in dollars) of a stock at time t, denoted by $S(t)$, may be viewed as a positive random variable on a probability space Ω. Now, consider the set Ω of three feasible price movement scenarios ω_1, ω_2, and ω_3 such that the one-step returns are indicated in the table below.

Scenario	$K(1)$	$K(2)$	$K(3)$
ω_1	32%	17%	-3%
ω_2	32%	-3%	-3%
ω_3	17%	32%	32%

(a) Assuming that the value of the stock at time 0 is \$60 and that a dividend of \$1 is paid at the end of each time period. The return for that period is computed before the payment of the dividend. Find the possible stock prices in a three-step.

Solution. From the formula

$$K(n) = \frac{S(n) - S(n-1) + \text{div}(n)}{S(n-1)},$$

we obtain

$$S(n) = S(n-1)[1 + K(n)] - \text{div}(n).$$

Using this, we obtain the following table.

Scenario	$S(0)$	$1 + K(1)$	$S(1)$	$1 + K(2)$	$S(2)$	$1 + K(3)$	$S(3)$
w_1	60	1.32	78.2	1.17	90.49	0.97	86.78
w_2	60	1.32	78.2	0.97	74.85	0.97	71.61
w_3	60	1.17	69.2	1.32	90.34	1.32	118.45

(b) If the probabilities of the three scenarios are $P(\omega_1) = 0.45$, $P(\omega_2) = 0.39$, and $P(\omega_3) = 0.16$, find all expected returns.

Solution. We have

$$\mathbb{E}[K(1)] = (0.45)(32\%) + (0.39)(32\%) + (0.16)(17\%) = 0.296 = 29.6\%,$$
$$\mathbb{E}[K(2)] = (0.45)(17\%) + (0.39)(-3\%) + (0.16)(32\%) = 0.116 = 11.6\%,$$
$$\mathbb{E}[K(3)] = (0.45)(-3\%) + (0.39)(-3\%) + (0.16)(32\%) = 0.026 = 2.6\%$$

3. (20 points) Consider the following one-period model. There is a bond, a stock, and a put option. The one-period interest rate for the bond is 10%. The stock can either go up to \$140 or down to \$100. Suppose we know that the put option is worth \$20 if the stock goes down, and worth nothing otherwise. At time 0, the put option sells for \$10.

(a) Find the value of the stock at time 0, by the portfolio method.

Solution. We must form a portfolio which duplicates the stock, using the bond and the option. Let $A(t), S(t), C(t)$ denote the prices of the bond, stock, and the option at time t. Let x, y denote the number of shares of the bond and the option in our portfolio. We can assume that the bond is worth \$100 at time 0 and \$110 at time 1. Let $t = 1$ and set the value $V(t)$ of the portfolio equal to the value of the stock, $S(1)$. We arrive at the following set of equations,

$$\begin{aligned} 110x &= 140 \\ 110x + 20y &= 100 \end{aligned}$$

The first equation gives $x = 140/110 = 14/11$. Subtracting the first equation from the second, we get $20y = -40$ or $y = -2$. Therefore the price of the stock at time 0 is equal to the value of the portfolio at time 0, namely

$$S(0) = 100x + 10y = \frac{1400}{11} - 20 = 107.27$$

(b) Find the risk-neutral probability p_* that the stock goes up.

Solution. We must find a risk-neutral probability p_*, the probability that the stock goes up, so that the expected return on the option equals the return on the bond. The equation is

$$(1 - p_*)20 = E_*[C(1)] = (1 + r)C(0) = 1.1 \times 10 = 11$$

Solving, we find that $1 - p_* = 11/20$ so $p_* = 9/20$.

(c) On the basis of part (b), recalculate the value of the stock at time 0. This number should agree with your answer in part (a).

Solution. We have

$$S(0) = \frac{1}{1+r}(140p_* + 100(1 - p_*)) = \frac{100}{110} \times \left(140 \times \frac{9}{20} + 100 \times \frac{11}{20}\right) = 107.27$$

4. (20 points)

(a) Suppose that a bond pays \$100 at the end of each year, for 2 years. In addition, it pays \$1000 at the end of the 2-year period. If the interest rate is 2% over the whole period, how much is the bond worth initially?

Solution. Let $r = .02$, the interest rate. The value of the bond would be the present value of the income stream, namely

$$\text{PV} = \frac{100}{1.02} + \frac{1100}{(1.02)^2} = 1155.32$$

(b) Suppose that the bond were a perpetuity paying 100 a year, at the end of each year, forever. How much would the bond be worth initially?

Solution. Using the geometric formula, we have

$$\begin{aligned} \text{PV} &= \frac{100}{1+r} + \frac{100}{(1+r)^2} + \cdots \\ &= \frac{100}{(1+r)}\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots\right) \\ &= \frac{100}{(1+r)} \cdot \frac{1}{1 - \frac{1}{1+r}} \\ &= \frac{100}{(1+r)} \cdot \frac{1+r}{r} \\ &= \frac{100}{r} \end{aligned}$$

or you can just remember the final formula. Putting in $r = .02$, we find that

$$\text{PV} = \frac{100}{0.2} = 5000$$

(c) If the income stream were to last for 20 years, how much would the bond be worth?

Solution. The bond is equivalent to the difference of two perpetuities, the first starting immediately and the second starting in 20 years. Each is worth 5000. Bringing these numbers back into present value, we find

$$\text{PV} = 5000 - \frac{1}{(1.02)^{20}} \times 5000 = 1635.14$$

5. (20 points) Suppose a bond doubles in 10 years. Find the following interest rates, giving your answer correct to 4 decimal places.

(a) Find the continuously compounded interest rate.

Solution. We must solve for r in $e^{10r} = 2$. Taking natural logarithms, we find $10r = \ln 2$

$$r = \frac{\ln 2}{10} = .0693$$

(b) Find the annual rate of interest.

Solution. If R is the rate of interest with continuously compounding, then the annual rate r would be given by $1 + r = e^R$, so

$$r = e^R - 1 = .0718$$

using the results of part (a).

(c) Find the rate of interest with compounding twice a year.

Solution. We would do the same as in part (b), but putting in $1/2$ to get $1 + r = e^{R/2}$, so

$$r = e^{R/2} - 1 = .0353$$