*********!!!!!!!!!!!!!!!! The midterm date: November 3, 2009 (Tuesday) !!!!!!!!!!!!!!!!*********

!!!!!!!!!!!!!!!!********* The midterm will be in HH 141, Starting time: 8am *********!!!!!!!!!!!!!!!!

——————————SOLUTIONS——————————

165 f09	Sample Midterm	Exam Time: Tu. 11/3/09, 8:00 - 9:30
Name:		Student No.:

Circle the name of your instructor: Prof. Salur (MW 2pm) Prof. Arikan (MWF 10am)

Instructions:

- Answer ALL questions from Section A
- You may use a handwritten sheet of notes. Calculators are NOT permitted.
- Read all questions carefully
- Unless explicitly told otherwise, you should explain all your answers fully.
- Do NOT seperate the pages of your exam.

Problem	Points	Score
A1	16	
A2	16	
A3	16	
A4	12	
A5	12	
A6	12	
A7	16	
Total	100	

Name:

Section A: Answer ALL questions.

Problem A1: [16=8+8 pts] Consider the family of curves given by the equation $y^2 = 2x + c$ where c is a constant, and x, y are variables.

(**a**) What is the differential equation which gives the slope of the family at a point (x, y)?

Solution:

If we implicitly dfferentiate the equation $y^2 = 2x + c$, we get $2y\dfrac{dy}{dx} = 2$. Solving for dy/dx we get

$$\frac{dy}{dx} = \frac{1}{y}, \quad (y \neq 0).$$

$$\boxed{\frac{dy}{dx} = \frac{1}{y}, \quad (y \neq 0)}$$

(**b**) Find the the orthogonal trajectories to the family given above. Give your answer in explicit form.

Solution:

The differential equation the orthogonal family satisfies is

$$\frac{dy}{dx} = -y.$$

This separates to

$$\frac{1}{y}dy = -dx.$$

Integrating both sides yields

$$\ln|y| = -x + c$$

and exponentiating gives

$$y = Ae^{-x}$$

where $A = \pm e^c$.

$$\boxed{y = Ae^{-x}}$$

Name:

Problem A2: [16=8+8 pts]

(**a**) Find the general solution of the equation

$$\frac{dy}{dx} - (\tan x)y = 8\sin^3 x.$$

Solution:

This is a first order linear equation. Since

$$\int -\tan(x)dx = -\ln|\sec(x)| + constant$$

The integrating factor can be taken as

$$I(x) = e^{-\ln(\sec(x))} = \frac{1}{\sec(x)} = \cos(x).$$

The equation can then be re-written as

$$(y\cos(x))' = 8\sin^3 x\cos x.$$

Integrating both sides yields

$$y\cos x = 2\sin^4 x + C.$$

Therefore, the general solution is

$$y = 2\sin^4 x\sec x + C\sec x.$$

$$\boxed{y = 2\sin^4 x\sec x + C\sec x}$$

(**b**) Solve the initial value problem

$$y' = y^3\sin x, \qquad y(0) = 1.$$

Solution:

This is separable equation, that separates to

$$\frac{1}{y^3}dy = \sin x dx.$$

Integrating both sides yields

$$-\frac{1}{2y^2} = -\cos x + C.$$

Since $y(0) = 1$, we see that $-\frac{1}{2} = -1 + C$ and so $C = \frac{1}{2}$. Thus the solution is

$$y^2 = \frac{1}{2\cos x - 1}.$$

$$\boxed{y^2 = \frac{1}{2\cos x - 1}}$$

Name:

Problem A3: [16=6+10 pts]

(**a**) For which value(s) of k is the following system of linear equations consistent?

$$\begin{aligned} x + 2y &= 1 \\ 2x + ky &= -5 \end{aligned}$$

Solution:

We construct the augmented matrix and row reduce

$$\begin{pmatrix} 1 & 2 & | & 1 \\ 2 & k & | & -5 \end{pmatrix} \overset{A_{12}(-2)}{\sim} \begin{pmatrix} 1 & 2 & | & 1 \\ 0 & k-4 & | & -7 \end{pmatrix}$$

The system is inconsistent if the bottom row of reduction is $\begin{pmatrix} 0 & 0 & | & -7 \end{pmatrix}$, i.e. if $k = 4$. Therefore, the system is consistent for any $k \neq 4$.

k can be any real except 4

(**b**) Find the solution(s) to the following system of linear equations:

$$\begin{aligned} 3x + y + 5z &= 5 \\ x + y - z &= 1 \\ 2x + y + 2z &= 3 \end{aligned}$$

Write your answer in vector format. If the system is inconsistent, write "inconsistent". Justify your answer. (No credit will be given for a method not based on either Gaussian or Gauss-Jordan elimination)

Solution:

We construct the augmented matrix and row reduce

$$\begin{pmatrix} 3 & 1 & 5 & | & 5 \\ 1 & 1 & -1 & | & 1 \\ 2 & 1 & 2 & | & 3 \end{pmatrix} \overset{P_{12}}{\sim} \begin{pmatrix} 1 & 1 & -1 & | & 1 \\ 3 & 1 & 5 & | & 5 \\ 2 & 1 & 2 & | & 3 \end{pmatrix} \overset{A_{12}(-3), A_{13}(-2)}{\sim} \begin{pmatrix} 1 & 1 & -1 & | & 1 \\ 0 & -2 & 8 & | & 2 \\ 0 & -1 & 4 & | & 1 \end{pmatrix}$$

$$\overset{M_2(-1/2)}{\sim} \begin{pmatrix} 1 & 1 & -1 & | & 1 \\ 0 & 1 & -4 & | & -1 \\ 0 & -1 & 4 & | & 1 \end{pmatrix} \overset{A_{23}(1)}{\sim} \begin{pmatrix} 1 & 1 & -1 & | & 1 \\ 0 & 1 & -4 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

This is consistent and reduces to the system

$$\begin{aligned} x + y - 4z &= 1 \\ y - 4z &= -1 \\ z &= t \end{aligned}$$

where t is a parameter. We compute $y = -1 + 4z = -1 + 4t, \quad x = 1 + z - y = 2 - 3t$. Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} t$$

Name:

Problem A4: [12=5+5+2 pts] Consider the matrix

$$A = \begin{pmatrix} -2 & 3 & -1 \\ 2 & 1 & 5 \\ 0 & 2 & 3 \end{pmatrix}.$$

(**a**) Compute $adj(A)$.

Solution:

The matrices of minors and cofactors are

$$M = \begin{pmatrix} -7 & 6 & 4 \\ 11 & -6 & -4 \\ 16 & -8 & -8 \end{pmatrix} \qquad C = \begin{pmatrix} -7 & -6 & 4 \\ -11 & -6 & 4 \\ 16 & 8 & -8 \end{pmatrix}.$$

The adjoint matrix $adj(A)$ is the transpose of C. Hence,

$$adj(A) = \begin{pmatrix} -7 & -11 & 16 \\ -6 & -6 & 8 \\ 4 & 4 & -8 \end{pmatrix}$$

(**b**) Compute $det(A)$.

Solution:

$det(A)$ can be computed using a cofactor expansion with respect to any row or column.
Let's expand with respect to the 1^{st} row of A:
Using the 1^{st} row of A and the 1^{st} row of C (corresponding cofactors), we get

$$det(A) = -2(-7) + 3(-6) + (-1)(4) = -8$$

$$det(A) = \quad -8$$

(**c**) Is A invertible? If so, find A^{-1}, if not justify your answer.

Solution:

As $det(A) \neq 0$, A is invertible. We get A^{-1} using the formula $A^{-1} = \dfrac{1}{det(A)}\, adj(A)$.

$$A^{-1} = -\frac{1}{8} \begin{pmatrix} -7 & -11 & 16 \\ -6 & -6 & 8 \\ 4 & 4 & -8 \end{pmatrix}$$

Name:

Problem A5: [12=4+4+4 pts] Suppose that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$$

(**a**) Find $\begin{vmatrix} a & b & c \\ 2g+d & 2h+e & 2i+f \\ g & h & i \end{vmatrix}$

Solution:

This matrix is obtained by adding 2 times row 1 to row 3.
This does not change the determinant.

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(**b**) Find $\begin{vmatrix} g & h & i \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix}$

Solution:

This matrix is obtained by swapping rows 1 and 3 and then multiplying row 3 by 2.
This multiplies the determinant by -1 and then by 2.

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(**c**) Find $\begin{vmatrix} 2a & d & g \\ 2b & e & h \\ 2c & f & i \end{vmatrix}$

Solution:

This matrix is obtained by multiplying row 1 by 2 and then transposing the matrix.
This multiplies the determinant by 2 with no change from the transpose.

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Name:

Problem A6: [12=4+4+4 pts] For each set of vectors, decide whether or not the set is linearly independent. Fully justify your answers.

(**a**) $\{e^t, e^{-t}, t\}$

Solution:

We compute the Wronskian: $W(e^t, e^{-t}, t) = \begin{vmatrix} e^t & e^{-t} & t \\ e^t & -e^{-t} & 1 \\ e^t & e^{-t} & 0 \end{vmatrix} = t(2) - 1(0) + 0 = 2t$ where we used the cofactor expansion along the 3rd column. Since $W(e^t, e^{-t}, t) \neq 0$ for $t \neq 0$, the functions are linearly independent.

linearly independent

(**b**) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \right\}$

Solution:

EITHER compute

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -4 \\ 3 & 2 & 1 \end{vmatrix} = 1(-1+8) - 1(2+12) + 1(4+3) = 0$$

so the vectors are linearly dependent.

OR use the augmented matrix of the system $x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 2 & -1 & -4 & | & 0 \\ 3 & 2 & 1 & | & 0 \end{pmatrix} \overset{A_{12}(-2), A_{13}(-3)}{\sim} \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & -3 & -6 & | & 0 \\ 0 & -1 & -2 & | & 0 \end{pmatrix} \overset{M_2(-1/3), A_{23}(1)}{\sim} \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

So, for any real t, we have a solution of the form $(x_1, x_2, x_3) = (t, -2t, t)$.
Thus, for any $t \neq 0$, there is a non-trivial linear dependency between the given column vectors:

$$t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

linearly dependent

(**c**) $\left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \right\}$

Solution:

The vector space $M_{2\times 2}(\mathbb{R})$ is known to have dimension 4.
As the set contains more than 4 vectors, it must be linearly dependent.

linearly dependent

Name:

Problem A7: [16=8+8 pts]

(**a**) Is $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$? Justify your answers.

Solution:

As $\mathbb{R}^3$ is 3 dimensional, this set is a basis if it is linearly independent or it spans $\mathbb{R}^3$.
Use one of the following methods:
Method-1:

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 1(1-2) - 3(-2-1) + 1(4+1) = -1 + 9 + 5 = 13 \neq 0$$

so the vectors are linearly independent ($\Rightarrow B$ is a basis).
Method-2:

$$\begin{pmatrix} 1 & 3 & 1 & x \\ 2 & -1 & 1 & y \\ 1 & 2 & -1 & z \end{pmatrix} \overset{A_{12}(-2), A_{13}(-1)}{\sim} \begin{pmatrix} 1 & 3 & 1 & x \\ 0 & -7 & -1 & y-2x \\ 0 & -1 & -2 & z-x \end{pmatrix} \overset{M_2(-1/7), A_{23}(1)}{\sim} \begin{pmatrix} 1 & 3 & 1 & x \\ 0 & 1 & 1/7 & (x-2y)/7 \\ 0 & 0 & -13/7 & z-x+(x-2y)/7 \end{pmatrix}$$

so the system is consistent for all x, y, z. Therefore, the vectors span $\mathbb{R}^3$ ($\Rightarrow B$ is a basis).
Method-3:

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix} \overset{A_{12}(-2), A_{13}(-1)}{\sim} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{pmatrix} \overset{M_2(-1/7), A_{23}(1)}{\sim} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1/7 & 0 \\ 0 & 0 & -13/7 & 0 \end{pmatrix}$$

so the only linear dependency between the vectors is trivial and they are linearly independent ($\Rightarrow B$ is a basis).

Yes

(**b**) Determine a basis for the subspace S of $\mathbb{R}^4$ consisting of all solutions to the linear system

$$x_1 - 2x_2 + 3x_3 + x_4 = 0$$
$$-2x_1 + 3x_2 - 4x_3 + x_4 = 0$$

What is the dimension, $dim(S)$, of S?

Solution:

We row reduce the coefficient matrix of the system to determine the system parameters and the solutions:

$$\begin{pmatrix} 1 & -2 & 3 & 1 \\ -2 & 3 & -4 & 1 \end{pmatrix} \overset{A_{12}(2)}{\sim} \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix} \overset{M_2(-1)}{\sim} \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & -3 \end{pmatrix} \overset{A_{21}(2)}{\sim} \begin{pmatrix} 1 & 0 & -1 & -5 \\ 0 & 1 & -2 & -3 \end{pmatrix}$$

Any solution is of the form $(x_1, x_2, x_3, x_4) = (s + 5t, 2s + 3t, s, t)$ where s, t are arbitrary reals (parameters). So $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 5 \\ 3 \\ 0 \\ 1 \end{pmatrix} t \;\middle|\; s, t \in \mathbb{R} \right\}$. Since the set $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly independent and spans S, it is a basis for S. The dimension of a subspace is the number of vectors in any basis of the subspace. Therefore, $dim(S) = 2$.

Basis=$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$, $dim(S) = 2$