######################SOLUTIONS######################

165 f09	Midterm	Exam Time: Tu. 11/3/09, 8:00 - 9:30
Name:		Student No.:

Circle the name of your instructor: Prof. Salur (MW 2pm) Prof. Arikan (MWF 10am)

Instructions:

- Answer ALL questions from Section A
- You may use a handwritten sheet of notes. Calculators are NOT permitted.
- Read all questions carefully
- Unless explicitly told otherwise, you should explain all your answers fully.
- Do NOT seperate the pages of your exam.

Problem	Points	Score
A1	16	
A2	14	
A3	16	
A4	12	
A5	12	
A6	16	
A7	14	
Total	100	

Name:

Section A: Answer ALL questions.

Problem A1: [16=8+8 pts]

(**a**) Solve the initial value problem

$$(1+x^2)y' + 1 = -y^2, \qquad y(0) = \sqrt{3}.$$

Solution:

This is a separable equation, that separates to

$$\frac{1}{1+y^2}\,dy = -\frac{1}{1+x^2}\,dx.$$

Integrating both sides yields

$$\tan^{-1}(y) = -\tan^{-1}(x) + C.$$

Using $y(0)=\sqrt{3}$, we compute that $\tan^{-1}(\sqrt{3}) = -\tan^{-1}(0) + C \Longrightarrow C = \frac{\pi}{3}$. Thus, the solution is

$$\tan^{-1}(y) = -\tan^{-1}(x) + \frac{\pi}{3}.$$

$$\boxed{\tan^{-1}(y) = -\tan^{-1}(x) + \frac{\pi}{3}}$$

(**b**) Find the general solution of the equation

$$\frac{dy}{dx} - x^{-1}y = 2\ln x.$$

Solution:

This is a first order linear equation. Note that $x > 0$ for the equation to make sense. Since

$$\int -\frac{1}{x}dx = -\ln|x| + constant = -\ln x + constant,$$

the integrating factor can be taken as

$$I(x) = e^{-\ln x} = \frac{1}{x}.$$

The equation can then be re-written as

$$\left(y\frac{1}{x}\right)' = 2\frac{\ln x}{x}.$$

Integrating both sides yields (use the substitution $u = \ln x$ to integrate right hand side)

$$y\frac{1}{x} = (\ln x)^2 + C.$$

Therefore, the general solution is

$$y = x(\ln x)^2 + Cx.$$

$$\boxed{y = x(\ln x)^2 + Cx}$$

Name:

Problem A2: [14=7+7 pts] Consider the family of curves given by the equation

$$5x^2 + e^{2y} + c = -2x$$

where c is a constant, and x, y are variables.

(**a**) What is the differential equation which gives the slope of the family at a point (x, y)?

Solution:

If we implicitly differentiate the equation $5x^2 + e^{2y} + c = -2x$ with respect to x, we get

$$10x + 2e^{2y}\frac{dy}{dx} = -2.$$

Solving for dy/dx we get

$$\frac{dy}{dx} = \frac{-2 - 10x}{2e^{2y}} = -\frac{1 + 5x}{e^{2y}}.$$

$$\boxed{\frac{dy}{dx} = -\frac{1 + 5x}{e^{2y}}}$$

(**b**) Find an equation for the orthogonal trajectories to the family given above.
(You may give your equation in the implicit form, i.e., you do not have to solve it for x or y.)

Solution:

The differential equation the orthogonal family satisfies is

$$\frac{dy}{dx} = \frac{e^{2y}}{1 + 5x}.$$

This separates to

$$e^{-2y}dy = \frac{1}{1 + 5x}\,dx.$$

Integrating both sides, we get

$$-\frac{1}{2}\,e^{-2y} = \frac{1}{5}\ln|1 + 5x| + c$$

which describes the orthogonal trajectories to the family $5x^2 + e^{2y} + c = -2x$.

$$\boxed{-\frac{1}{2}\,e^{-2y} = \frac{1}{5}\ln|1 + 5x| + c}$$

Name:

Problem A3: [16=8+8 pts]

(**a**) Find the solution(s) to the following system of linear equations:

$$\begin{aligned} x + 5z &= 0 \\ 3x - 2y + 11z &= 2 \\ 2x - 2y + 6z &= 2 \end{aligned}$$

Write your answer in vector format. If the system is inconsistent, write "inconsistent". Justify your answer. (No credit will be given for a method not based on either Gaussian or Gauss-Jordan elimination)

Solution:

We construct the augmented matrix and row reduce

$$\begin{pmatrix} 1 & 0 & 5 & | & 0 \\ 3 & -2 & 11 & | & 2 \\ 2 & -2 & 6 & | & 2 \end{pmatrix} \overset{A_{12}(-3),A_{13}(-2)}{\sim} \begin{pmatrix} 1 & 0 & 5 & | & 0 \\ 0 & -2 & -4 & | & 2 \\ 0 & -2 & -4 & | & 2 \end{pmatrix} \overset{M_2(-1/2),A_{23}(2)}{\sim} \begin{pmatrix} 1 & 0 & 5 & | & 0 \\ 0 & 1 & 2 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

This is consistent and reduces to the system

$$\begin{aligned} x + 5z &= 0 \\ y + 2z &= -1 \\ z &= t \end{aligned}$$

where t is a parameter. We compute $y = -1 - 2z = -1 - 2t, \quad x = -5z = -5t$. Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix} t$$

(**b**) For which value(s) of k is the following system of linear equations inconsistent?

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 5y + z &= 7 \\ x + y - 4z &= -k \end{aligned}$$

Solution:

We construct the augmented matrix and row reduce

$$\begin{pmatrix} 1 & 2 & -1 & | & 3 \\ 2 & 5 & 1 & | & 7 \\ 1 & 1 & -4 & | & -k \end{pmatrix} \overset{A_{12}(-2),A_{13}(-1)}{\sim} \begin{pmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & 3 & | & 1 \\ 0 & -1 & -3 & | & -3-k \end{pmatrix} \overset{A_{23}(1)}{\sim} \begin{pmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 0 & | & -2-k \end{pmatrix}$$

So the system is consistent if the bottom row $\begin{pmatrix} 0 & 0 & 0 & | & -2-k \end{pmatrix}$ of reduction is a zero row, i.e. if $k = -2$. Therefore, the system is inconsistent for any $k \neq -2$.

k is any real except -2

Name:

Problem A4: [12=4+4+4 pts] Suppose that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

(**a**) Find $\begin{vmatrix} a & 3d & a+g \\ b & 3e & b+h \\ c & 3f & c+i \end{vmatrix}$

Solution:

This matrix is obtained by multiplying row 2 by 3, adding row 1 to row 3, and then transposing the matrix. This multiplies the determinant by 3 with no change from adding rows and taking transpose.

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(**b**) Find $\begin{vmatrix} g & h & i \\ 5a & 5b & 5c \\ d & e & f \end{vmatrix}$

Solution:

This matrix is obtained by multiplying row 1 by 5, then swapping rows 1 and 3 and then swapping rows 2 and 3 . This multiplies the determinant by 5 and by -1 and then by another -1.

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(**c**) Find $\begin{vmatrix} g & h & i \\ d+5g & e+5h & f+5i \\ a & b & c \end{vmatrix}$

Solution:

This matrix is obtained by adding 5 times row 3 to row 2 and then swapping rows 1 and 3.
Adding rows does not change the determinant, but swapping rows multiplies the determinant by -1.

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Name:

Problem A5: [12=5+5+2 pts] Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & 2 & 7 \\ 5 & 1 & -1 \end{pmatrix}.$$

(**a**) Compute $adj(A)$.

Solution:

The matrices of minors and cofactors are

$$M = \begin{pmatrix} -9 & -32 & -13 \\ -1 & -2 & -3 \\ 7 & 14 & 7 \end{pmatrix} \qquad C = \begin{pmatrix} -9 & 32 & -13 \\ 1 & -2 & 3 \\ 7 & -14 & 7 \end{pmatrix}.$$

The adjoint matrix $adj(A)$ is the transpose of C. Hence,

$$adj(A) = \begin{pmatrix} -9 & 1 & 7 \\ 32 & -2 & -14 \\ -13 & 3 & 7 \end{pmatrix}$$

(**b**) Compute $det(A)$.

Solution:

$det(A)$ can be computed using a cofactor expansion with respect to any row or column.
Let's expand with respect to the 1^{st} row of A:
Using the 1^{st} row of A and the 1^{st} row of C (corresponding cofactors), we get

$$det(A) = 2(-9) + 1(32) + 0(-13) = 14.$$

$$det(A) = \quad 14$$

(**c**) Is A invertible? If so, find A^{-1}, if not justify your answer.

Solution:

As $det(A) \neq 0$, A is invertible. We get A^{-1} using the formula $A^{-1} = \dfrac{1}{det(A)}\, adj(A)$.

$$A^{-1} = \frac{1}{14} \begin{pmatrix} -9 & 1 & 7 \\ 32 & -2 & -14 \\ -13 & 3 & 7 \end{pmatrix}$$

Name:

Problem A6: [16=8+8 pts]

(**a**) Find a basis for the subspace S of $\mathbb{R}^4$ consisting of all solutions to the linear system

$$\begin{aligned} x_1 - 2x_2 + 4x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 - 2x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 - 2x_3 + x_4 &= 0 \end{aligned}$$

What is the dimension of S?

Solution:

We row reduce the coefficient matrix of the system to determine the system parameters and the solutions:

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & -2 & 2 \\ 1 & 2 & -2 & 1 \end{pmatrix} \underset{A_{13}(-1)}{\overset{A_{12}(-2)}{\sim}} \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 7 & -10 & 0 \\ 0 & 4 & -6 & 0 \end{pmatrix} \underset{A_{23}(-4)}{\overset{M_2(1/7)}{\sim}} \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & -10/7 & 0 \\ 0 & 0 & -2/7 & 0 \end{pmatrix} \underset{\substack{A_{31}(-4) \\ A_{32}(10/7)}}{\overset{M_3(-2/7)}{\sim}} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Any solution is of the form $(x_1, x_2, x_3, x_4) = (-t, 0, 0, t)$ where t is arbitrary real (parameter). So

$$S = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}. \text{ Since the set } \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is linearly independent and spans } S,$$

it is a basis for S. The dimension of a subspace is the number of vectors in any basis of the subspace. Therefore, $dim(S) = 1$.

Basis= $\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, $dim(S) = 1$

(**b**) Is $B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 8 \\ 6 \\ 0 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$? Justify your answers.

Solution:

As $\mathbb{R}^3$ is 3 dimensional, this set is a basis if it is linearly independent or it spans $\mathbb{R}^3$.

We check linear (in)dependency by using determinant:

$$\begin{vmatrix} 1 & 2 & 8 \\ -1 & 5 & 6 \\ 1 & -2 & 0 \end{vmatrix} = 8(2-5) - 6(-2-2) + 0(5+2) = -24 + 24 + 0 = 0$$

where we used the cofactor expansion along the 3rd column. So the vectors are linearly dependent
($\Rightarrow B$ is not a basis).

No

Name:

Problem A7: [14=5+5+4 pts] For each set of vectors, decide whether or not the set is linearly independent. Fully justify your answers.

(**a**) $\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$

Solution:

Row-reduce the augmented matrix of the system $c_1 \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 1 & 2 & |0 \\ -1 & -1 & -1 & |0 \\ -1 & 2 & 1 & |0 \\ -1 & 3 & -1 & |0 \end{pmatrix} \underset{A_{14}(1),M_3(1/3)}{\overset{A_{12}(1),A_{13}(1)}{\sim}} \begin{pmatrix} 1 & 1 & 2 & |0 \\ 0 & 0 & 1 & |0 \\ 0 & 1 & 1 & |0 \\ 0 & 4 & 1 & |0 \end{pmatrix} \underset{A_{24}(-1),P_{23}}{\overset{A_{21}(-2),A_{23}(-1)}{\sim}} \begin{pmatrix} 1 & 1 & 0 & |0 \\ 0 & 1 & 0 & |0 \\ 0 & 0 & 1 & |0 \\ 0 & 4 & 0 & |0 \end{pmatrix} \underset{A_{24}(-4)}{\overset{A_{21}(-1)}{\sim}} \begin{pmatrix} 1 & 0 & 0 & |0 \\ 0 & 1 & 0 & |0 \\ 0 & 0 & 1 & |0 \\ 0 & 0 & 0 & |0 \end{pmatrix}$$

So, the system has only the trivial solution $(c_1, c_2, c_3) = (0, 0, 0)$.
Thus, the vectors are linearly independent.

linearly independent

(**b**) $\{e^{3x},\ e^{5x},\ e^{2x}\}$

Solution:

We compute the Wronskian:

$$W(e^{3x}, e^{5x}, e^{2x}) = \begin{vmatrix} e^{3x} & e^{5x} & e^{2x} \\ 3e^{3x} & 5e^{5x} & 2e^{2x} \\ 9e^{3x} & 25e^{5x} & 4e^{2x} \end{vmatrix} = e^{3x}(20e^{7x} - 50e^{7x}) - e^{5x}(12e^{5x} - 18e^{5x}) + e^{2x}(75e^{8x} - 45e^{8x}) = 6e^{10x}$$

where we used the cofactor expansion along the 1st row. Since $W(e^{3x}, e^{5x}, e^{2x}) = 6e^{10x} \neq 0$ for any x, the functions are linearly independent.

linearly independent

(**c**) $\{x - 1,\ 2 - 8x^2,\ 5x^2 + 7x,\ 6 - 3x + x^2\}$

Solution:

The vector space $P_2(\mathbb{R})$ is known to have dimension 3. (For instance, $\{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$.)
As the set contains more than 3 vectors, it must be linearly dependent.

linearly dependent