Math 165: Linear Algebra/Diff. Eq.
Midterm 2
November 8, 2012

NAME (please print legibly): _______________________________________________________
Student ID Number: ____________________________________________________________

Indicate your instructor with a check in the box:

<table>
<thead>
<tr>
<th>Mark Herman</th>
<th>MWF 10:00 - 10:50 AM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Giorgis Petridis</td>
<td>MWF 12:00 - 12:50 PM</td>
</tr>
<tr>
<td>David Karapetyan</td>
<td>MW 2:00 - 3:15 PM</td>
</tr>
</tbody>
</table>

- There are no notes, textbooks, etc. allowed on this exam. The presence of
  calculators, cell phones, iPods and other electronic devices at this exam
  is strictly forbidden. Having notes, calculators, or electronic devices where
  they are visible to you will be considered academic dishonesty

- Show your work and justify your answers. You may not receive full credit
  for a correct answer if insufficient work is shown or insufficient justification
  is given.

- Clearly circle or label your final answers.
1. (19 pts) Evaluate the determinant of the following matrix $A$ by completing two steps as instructed.

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & -1 & -1 \\ 1 & -1 & 3 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

(i) (4 points) Use row operations to show that

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & -1 & -1 \\ 1 & -1 & 3 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix} = -4 \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 3 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix}.$$

(ii) (9 points) Use a cofactor expansion to evaluate the determinant appearing on the right hand side and deduce the value of the original determinant $\det(A)$ from part (i).
(iii) (6 points) Suppose $B$ is a $4 \times 4$ matrix satisfying $AB = 2I_4$. Use your answer from the previous part to calculate the determinant of $B$.

2. (6 pts) Let $C \in M_n(\mathbb{R})$ be an $n \times n$ real matrix. Show that $\det(CC^T) \geq 0$. 
3. (15 pts) You are given a vector space $V$ and a subset $U \subseteq V$. Is $U$ a subspace of $V$? Justify your answer briefly.

(i) (5 points) $V = C^2(\mathbb{R})$ is the set of all real valued functions that have continuous second derivatives on $\mathbb{R}$ and $U = \{y \in C^2(\mathbb{R}) : y'' + 3y' + 7y = 0\}$.

(ii) (5 points) $V = \mathbb{R}^3$ and $U = \{(x, y, z) \in \mathbb{R}^3 : x = 2y + z\}$.
(iii) (5 points) \( V = \mathbb{R}^3 \) and \( U = \{ \mathbf{u} \in \mathbb{R}^3 : \mathbf{u} \cdot \mathbf{u} \leq 1 \} \). As usual \( \mathbf{u} \cdot \mathbf{u} \) is the dot product of \( \mathbf{u} \) with itself: if \( \mathbf{u} = (u_1, u_2, u_3) \), then \( \mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2 \).
4. (15 pts) In the following examples you are given a vector space $V$ and a set of vectors $S \subseteq V$. Is $S$ a *spanning set* of $V$? Justify your answer.

(i) (5 points) $V = \mathbb{R}^3$ and $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ \end{pmatrix} \right\}$. 

(ii) (5 points) $V = P_2$ the space of real polynomials of degree at most two and 

$S = \{ f_1(t) = 1, f_2(t) = 1 + t, f_3(t) = t + t^2 \}$. 


(iii) (5 points) \( V = \{ A \in M_2(\mathbb{R}) : A^T = A \} \) is the space of \( 2 \times 2 \) real symmetric matrices and

\[
S = \left\{ M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.
\]
5. (15 pts) You are given a vector space $V$ and a set of vectors $S \subseteq V$. Is $S$ a *linearly dependent* or a *linearly independent* set? Justify your answer briefly.

(i) (5 points) $V = \mathbb{R}^3$ and

$$S = \left\{ v_1 = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

(ii) (5 points) $V$ is the space of real valued functions and

$$S = \{ f_1(t) = (1 + t)^2, f_2(t) = t, f_3(t) = t^2, f_4(t) = 1 \}.$$
(iii) (5 points) $V$ is the space of real valued functions and

$$S = \{ f_1(t) = e^t, f_2(t) = \sin(t), f_3(t) = 1 \}.$$
6. **(10 pts)** You are given a vector space $V$. Write down a basis for $V$ and determine $\dim(V)$. (no justification necessary)

(i) (5 points) $V$ is the space of $4 \times 4$ real diagonal matrices.

(ii) (5 points) $V = \{f \in P_4 : f(0) = 0\}$ is the space of real polynomials of degree at most four that satisfy $f(0) = 0$. 
7. (20 pts) Let \( A \) be the matrix \[
\begin{bmatrix}
1 & 1 & 3 & 2 & 5 \\
1 & 2 & 2 & 3 & 1
\end{bmatrix}
\]. Define the nullspace of \( A \), find a basis for it, and clearly explain how you know it is a basis.