MATH 165
MIDTERM 1 EXAM
Oct 18, 2012

NAME (please print legibly): ________________
Student ID Number: _______________________

- The Examination’s duration is one hour and 15 minutes.
- No calculators or notes are allowed on this exam.
- Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown. Attempt all questions.
- Please circle your Instructor: Petridis (MWF 12-12:50)

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1. **(16 pts)** Solve the following two initial value problems. \( y' \) stands for \( \frac{dy}{dx} \).

(i) \( y' = (1 - 2y) \), \( y(0) = 0 \). Find the value of \( y \) as \( x \to \infty \).

\[
\frac{dy}{dx} = \frac{1 - 2y}{1 - 2y}
\]

So
\[
\int \frac{dy}{1 - 2y} = \int dx
\]

Integrating gives
\[
\ln |1 - 2y| = -2x + C
\]

Therefore
\[
1 - 2y = Ce^{-2x}
\]

Using the initial condition \( y(0) = 0 \), gives \( C = 1 \).

The solution is
\[
y = \frac{1 - e^{-2x}}{2}
\]

and, as \( x \to \infty \), \( y \to \frac{1}{2} \).

(ii) \( y' + \frac{y}{x} = 1, \ y(1) = 2 \).

The equation is in the wrong form so we simply multiply both sides with the integrating factor \( T(x) = e^\int \frac{dx}{x} = \frac{e}{x} \).

\[
x \cdot \frac{y'}{x} + \frac{y}{x} = 1
\]

\[
\iff (xy)' = x
\]

Integrating gives
\[
xy = \int x \, dx = \frac{x^2}{2} + C \quad \text{and so}
\]

\[
y = \frac{x^2 + C}{x}
\]

The initial condition gives
\[
C = 3 \quad \text{and therefore} \quad \sqrt{y} = \sqrt{\frac{x^2 + 3}{x}}
\]
2. (16 pts) Find the general solution to the following two differential equations.

(i) $y' + \cos(x)\ y = 0$.

We have \[ \frac{dy}{dx} = -\cos(x)\ y \quad \Rightarrow \quad \frac{dy}{y} = -\cos(x)\ dx. \]

Integrating gives: \[ \int \frac{dy}{y} = \int -\cos(x)\ dx \quad \Rightarrow \]

\[ \ln|y| = -\sin(x) + C \quad \Rightarrow \]

\[ y = C\ e^{-\sin(x)} \]

(ii) $y' + y = e^{-x}$.

The equation is in the correct form. So we multiply both sides by the integrating factor \( I(x) = e^{\int dx} = e^x \):

\[ e^x y' + e^x y = 1 \quad \Rightarrow \quad (e^x y)' = 1 \]

Integrating gives: \[ e^x \cdot y = x + C \]

\[ y = xe^{-x} + Ce^{-x} \]
3. (15 pts)

(i) (10 points) Find the unique solution to the following system of linear equations and verify that your answer is indeed the solution.

\[
\begin{align*}
    x + y + z &= 3 \\
    x - y - z &= -3 \\
    x + 2y - z &= 0
\end{align*}
\]

The system can be written as

\[
\begin{bmatrix}
1 & 1 & 1 & | & 3 \\
1 & -1 & -1 & | & -3 \\
1 & 2 & -1 & | & 0
\end{bmatrix}
\]

By Gaussian elimination:

\[
\begin{bmatrix}
1 & 1 & 1 & | & 3 \\
1 & -1 & -1 & | & -3 \\
1 & 2 & -1 & | & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 & | & 3 \\
0 & -2 & -2 & | & 6 \\
0 & 1 & -1 & | & -3
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 & | & 3 \\
0 & 1 & -1 & | & -2 \\
0 & 1 & -1 & | & -2
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 & | & 3 \\
0 & 1 & 0 & | & 4 \\
0 & 0 & 1 & | & 2
\end{bmatrix}
\]

Back substitution gives 2 = 2, 1 = 1, and 3 = 3.

This is the solution as:

\[
\begin{align*}
    x + y + z &= 3 \\
    x - y - z &= -3 \\
    x + 2y - z &= 0
\end{align*}
\]

(ii) (5 points) Find the rank of the matrix

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & -1 \\
1 & 2 & -1
\end{bmatrix}
\]

and briefly explain why your answer justifies that the system of linear equations in part (i) has a unique solution. You may refer to calculations you have done while answering part (i).

A row echelon form of the matrix is

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

So the rank of the row echelon matrix is 3 (not 2, as is the case), and hence the rank of the matrix is 3. Therefore, the 3x3 system has a unique solution as the rank of its coefficient matrix is 3.
4. (15 pts) Using methods developed in the course: (i) (10 points) Consider the following system of linear equations

\[
\begin{align*}
    x + y &= 1 \\
    y + z &= 1 \\
    x + 2y + z &= 2.
\end{align*}
\]

Verify that \(x = 1, y = 0, z = 1\) is a solution and find the set of solutions of the system in terms of a free variable \(t\).

\[
\begin{align*}
    x &= 1, \\
    y &= 0, \\
    z &= 1. 
\end{align*}
\]

The general solution is given by Gaussian elimination:

\[
\begin{bmatrix}
    1 & 1 & 0 \\
    1 & 2 & 1 \\
\end{bmatrix}
\to
\begin{bmatrix}
    1 & 1 & 0 \\
    0 & 0 & 1 \\
\end{bmatrix}
\to
\begin{bmatrix}
    1 & 1 & 0 \\
    0 & 0 & 1 \\
\end{bmatrix}
\]

Let \(z = t\) be the free variable. Back substitution gives:

\[
\begin{align*}
    x &= 1 - z \\
    y &= 1 - z - t \\
    z &= t
\end{align*}
\]

So the general solution is

\[
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix} = \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix} + t \begin{bmatrix}
    -1 \\
    -1 \\
    1
\end{bmatrix}
\]

(ii) (5 points) Find the rank of the matrix

\[
\begin{bmatrix}
    1 & 1 & 0 \\
    1 & 0 & 1 \\
    2 & 1 & 1
\end{bmatrix}
\]

and briefly explain why your answer justifies there is precisely one free variable in the set of solutions to the system in part (i). You may refer to calculations you have done while answering part (i).

A row-reduced form is

\[
\begin{bmatrix}
    1 & 1 & 0 \\
    0 & 0 & 1 \\
\end{bmatrix}
\]

so the rank of the matrix is \(2\), hence there is one free variable.

The number of free variables is \(2\) and the rank is \(1 \implies 2 - 1 = 1\).
5. (15 pts) Using methods developed in the course: (i) (10 points) Find the inverse of the matrix

\[
A = \begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & 1 \\
1 & 2 & 2 \\
\end{bmatrix}
\]

and verify that your answer is correct.

\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & 1 \\
1 & 2 & 2 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
1 & 2 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Indeed, \( A \cdot A^{-1} = \begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & 1 \\
1 & 2 & 2 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \)
(ii) (2 points) What does the existence of the inverse in parts (i) imply for the set of solutions to the system of linear equations
\[
\begin{cases}
  x + 2y + z = 0 \\
  2y + z = 0 \\
  x + 2y + 2z = 0.
\end{cases}
\]
The system has the unique solution \( \begin{cases} x = 0 \\
  y = 0 \\
  z = 0 \end{cases} \).
Because \( A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \), we have \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

6. (11 pts)
Let \( M(t) \) be the following matrix function:
\[
M(t) = \begin{bmatrix}
\cos(t) & -\sin(t) \\
\sin(t) & \cos(t)
\end{bmatrix}.
\]
Verify the following three properties of \( M(t) \):
(i) (4 points) \( M^{-1} = M^T \).

\[
M^T = M^{-1} \quad \text{because} \quad M^T M = \begin{bmatrix}
-\cos(t) & \sin(t) \\
-\sin(t) & -\cos(t)
\end{bmatrix} \begin{bmatrix}
-\cos(t) & -\sin(t) \\
\sin(t) & \cos(t)
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]
(ii) (2 points) $M^T M M^T = M^T$.

\[
M^T M M^T = (M^T M) M^T = M^T = I_2 \implies M^T = M^T.
\]

(iii) (5 points) $M(t)$ is the solution to the initial value problem: \( \frac{d^2 M}{dt^2} + M = 0 \) and $M(0) = I_2$.

\[
\frac{d^2}{dt^2} M(t) = \begin{bmatrix}
\frac{d^2}{dt^2} (\cos(t)) & \frac{d}{dt} (\sin(t)) \\
\frac{d^2}{dt^2} (\sin(t)) & \frac{d}{dt} (\cos(t))
\end{bmatrix} = \begin{bmatrix}
-\cos(t) & \sin(t) \\
-\sin(t) & -\cos(t)
\end{bmatrix} = -M(t)
\]

\[
\frac{d^2}{dt^2} M(1) + M(1) = 0.
\]

\[
M(0) = \begin{bmatrix}
\cos(0) & -\sin(0) \\
\sin(0) & \cos(0)
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = I_2.
\]
7. (12 pts) Using methods developed in the course: Determine whether the following systems of linear equations are consistent or not. If they are consistent also determine whether they have a unique solution or an infinite number of solutions. Explain briefly your reasoning.

\[
\begin{align*}
(i) \quad \begin{cases}
x + y + z &= 3 \\
y + z &= 2 \\
z &= 1.
\end{cases}
\end{align*}
\]

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

so the unique solution is \(x = y = z = 1\).

Uniqueness comes from the fact that \(\text{rank (matrix of coefficients)} = \# \text{unknowns}\).

Note. \(x = y = z = 1\) does satisfy the system. Always check your answer.

\[
(ii) \quad \begin{cases}
x + y + z &= 0 \\
y + z &= 0 \\
x + 2y + 2z &= 1.
\end{cases}
\]

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 2 & 2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

Inconsistent because of third row: \(000:1\) \(\#\).
\[
\begin{align*}
(x+y &= 1 \\
-1 &= 0 \\
x-z &= 1
\end{align*}
\]

So \( z = t \) is a free variable and by back substitution:
\[
\begin{align*}
y &= -2 - t \\
x &= 1 - y = 1 + t
\end{align*}
\]

So the system is consistent and has general solution:
\[
\begin{pmatrix}
x \\ y \\ t
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}
\]