Math 165: Linear Algebra/Diff. Eq.

Final Exam
December 16, 2012

NAME (please print legibly): ____________________________________________
Your University ID Number: ____________________________________________
Indicate your instructor with a check in the box:

<table>
<thead>
<tr>
<th>Instructor</th>
<th>Class Times</th>
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<tbody>
<tr>
<td>Mark Herman</td>
<td>MWF 10:00 - 10:50 AM</td>
</tr>
<tr>
<td>David Karapetyan</td>
<td>MW 2:00 - 3:15 PM</td>
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<tr>
<td>Giorgis Petridis</td>
<td>MWF 12:00 - 12:50 PM</td>
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- The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.

- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

- Clearly circle or label your final answers.

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<th>QUESTION</th>
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<td>TOTAL</td>
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1. (20 points)

(a) Solve the initial value problem \( xy^3 - \sqrt{1 + x^2} \frac{dy}{dx} = 0; \ y(0) = 2. \)

(b) Find the general solution of \( t \frac{dy}{dt} - 3y = t^6 e^t. \)
2. (20 points) Consider the matrix

\[ A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -3 \\ 2 & 3 & -3 \end{bmatrix} \]

(a) Find \( A^{-1} \).

(b) Use your answer from part (a) to solve the system

\[
\begin{align*}
x & + 2y - z = 1 \\
3x & + 7y - 3z = 1 \\
2x & + 3y - 3z = 1
\end{align*}
\]
3. (12 points)

(a) (6 pts) Which of the following are subspaces of $\mathbb{R}^3$? (CIRCLE ALL THAT APPLY- answer only, no partial credit. Your answer must be perfect to receive credit for this part.)

(i) $\{(x, y, z) : -x + 2z + y + 4 = 0\}$
(ii) $\{(a, 2a - b, -2a + 3b) : a, b \in \mathbb{R}\}$
(iii) $\{(x, y, z) : -x + 2y \geq 0\}$

(b) (6 pts) Which of the following are subspaces of $C^2(\mathbb{R})$ (the vector space of all real functions that have continuous second derivatives on $(-\infty, \infty)$)? (CIRCLE ALL THAT APPLY- answer only, no partial credit. Your answer must be perfect to receive credit for this part.)

(i) $\{y \in C^2(\mathbb{R}) : 10 \frac{d^2y}{dx^2} + 20 \frac{dy}{dx} + 30y = 0\}$
(ii) $\{y \in C^2(\mathbb{R}) : \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = \cos(x)\}$
(iii) $\{y \in C^2(\mathbb{R}) : y''(1) = 0\}$
4. (18 points)

(a) (6 pts) Which of the following are linearly INDEPENDENT subsets of $\mathbb{R}^3$? (CIRCLE ALL THAT APPLY- answer only, no partial credit. Your answer must be perfect to receive credit for this part.)

(i) \{ (2, -5, 4) \}
(ii) \{ (2, -5, 4), (-6, 15, -12) \}
(iii) \{ (2, -5, 4), (1, -2, 1), (-1, -4, 1) \}
(iv) \{ (1, 1, 0), (1, -1, 2), (1, 3, 1), (2, -5, 4) \}

(b) (6 pts) Which of the following are spanning sets of $P_2$ (the vector space of all real polynomials of degree 2 or less)? (CIRCLE ALL THAT APPLY- answer only, no partial credit. Your answer must be perfect to receive credit for this part.)

(i) \{ 2 - 5x + 4x^2 \}
(ii) \{ 2 - 5x + 4x^2, -6 + 15x - 12x^2 \}
(iii) \{ 2 - 5x + 4x^2, 1 - x, 4 - 7x + 4x^2 \}
(iv) \{ 1 - x, 1 + x, x^2, 2 - 5x + 4x^2 \}

(c) (6 pts) Which of the following are bases of $\mathbb{R}^3$? (CIRCLE ALL THAT APPLY- answer only, no partial credit. Your answer must be perfect to receive credit for this part.)

(i) \{ (1, -2, 1), (4, 2, -3) \}
(ii) \{ (1, 0, 1), (3, -2, 1), (-5, 6, 1) \}
(iii) \{ (1, -1, 0), (6, -2, 0), (10, 6, 1) \}
(iv) \{ (2, -1, 1), (4, 3, 0), (1, 0, 0), (5, 8, -3) \}
5. **(20 points)** If \( A = \begin{bmatrix} 1 & 2 & 5 & 5 & -10 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & -1 & -4 & -2 & 8 \\ 2 & 3 & 6 & 7 & -10 \end{bmatrix} \) then the reduced row echelon form of \( A \) is given by \( \begin{bmatrix} 1 & 0 & -3 & 0 & 8 \\ 0 & 1 & 4 & 0 & -4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \). Let \( T \) be the linear transformation given by \( T(\vec{x}) = A\vec{x} \).

(a) (2 pts) The range of \( T \) (also called image of \( T \)) is a subspace of \( \mathbb{R}^d \) for what value of \( d \)?

(b) (2 pts) The kernel of \( T \) is a subspace of \( \mathbb{R}^k \) for what value of \( k \)?

(c) (7 pts) Find a basis for the range of \( T \).

(d) (7 pts) Find a basis for the kernel of \( T \).

(e) (2 pts) What is the dimension of the kernel of \( T \)?
6. (20 points) Find the general solution of:

\[ y^{(5)} + 20y^{(4)} + 100y''' = 0 \]
7. (20 points) Solve the initial value problem:

\[ y'' + 6y' + 10y = 60 \cos(2t) , \quad y(0) = 5, \quad y'(0) = 3 \]
8. (5 points) Let $A$ be an $n \times n$ matrix and suppose 0 is an eigenvalue of $A$. Determine which of the following statements are TRUE (CIRCLE ALL THAT APPLY- answer only, no partial credit. Your answer must be perfect to receive credit for this part.)

(i) $\det(A) = 0$.
(ii) $A$ is invertible.
(iii) $Ax = 0$ has only the trivial solution.
(iv) $\text{rank}(A) < n$.

9. (10 points) Suppose $v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ are eigenvectors of a $2 \times 2$ matrix $A$ with corresponding eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$ respectively. Let $\bar{v} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$. Express $\bar{v}$ as a linear combination of $v_1$ and $v_2$ and find $A\bar{v}$. 

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10. (20 points) Let

\[ A = \begin{bmatrix} 5 & 12 & -4 \\ 0 & 1 & 0 \\ 2 & 6 & -1 \end{bmatrix}. \]

(a) Find the eigenvalues of \( A \).

(b) Find a basis and the dimension of the eigenspace for each eigenvalue found in part (a).

(c) Is the matrix defective? Why or why not?
11. **(10 points)** Find the general solution of the system of differential equations:

\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{bmatrix} = \begin{bmatrix}
1 & -5 \\
-2 & -2
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]