MATH 165 — FALL 2011 MIDTERM 2
November 17, 8:00-9:15AM

Name: Solutions

Student ID#:_______________________________________

Instructor (Circle One): PALSSON HERMAN

- Show all your work, use backs of pages if necessary. Points may be deducted for correct answers with no justification.

- No calculators, books, notes, etc. are allowed on this exam.

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1. (12 pts) Evaluate the following determinant:

\[
\begin{vmatrix}
1 & 2 & -1 & 0 \\
0 & -1 & 1 & 2 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 \\
\end{vmatrix}
\]

(cofactor along column 1): 

\[D = (1) \begin{vmatrix} -1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} + (1) \begin{vmatrix} 2 & -1 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}\]

\[= (2) \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix}\]


\[= 6 - 3 - 6 - 3 = -6\]
2. (10 pts) Let $A$ and $B$ be $3 \times 3$ matrices where $\det(A) = -2$ and $\det(B) = 5$. Evaluate the following determinants:

(a) $\det(2B^TA^3) = \det(2B^T) \det(A^3)$

Using $\det(CD) = \det(C) \det(D)$ for $C,D$ non-zero since $B^T$ is $3 \times 3$, $2B^T$ is obtained by multiplying 3 rows by 2.

$\det(B^T) = \det(B)$

$= \det(2B^T) \det(AA^2)$

$= \det(2B^T) \det(A) \det(A^2)$

$= 2^3 \det(B^T) \det(A) \det(A^2)$

$= 2^3 \det(B) \det(A) \det(A) \det(A)$

$= 2^3 \cdot 5 \cdot (-2)^3$

$= -2^6 \cdot 5 = -320$

(b) $\det(A^{-1}BA) = \det(A^{-1}) \det(BA)$

Since $\det(A^{-1}) = \frac{1}{\det(A)}$

$= \frac{1}{\det(A)} \det(B) \det(A)$

$= \frac{\det(B) \det(A)}{\det(A)}$

$= \det(B)$

$= 5$
3. (12 pts)

(a) Is \( \{(x, y, z) \mid 2x - y + 3z = 0 \} \) a subspace of \( \mathbb{R}^3 \)?

Yes: \( \{(x_1, y_1, z_1) \in \mathbb{R}^3 \mid 2x_1 - y_1 + 3z_1 = 0 \} \)

\( \{(x_2, y_2, z_2) \in \mathbb{R}^3 \mid 2x_2 - y_2 + 3z_2 = 0 \} \)

Closure of \( \{(x_1, y_1, z_1) \in \mathbb{R}^3 \mid 2x_1 - y_1 + 3z_1 = 0 \} \):

\[ (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \]

and \( 2(x_1, y_1, z_1) = (2x_1, 2y_1, 2z_1) \)

(b) Is \( \{(x_1, x_2, x_3, x_4) \mid x_1 - x_2 \geq 0 \} \) a subspace of \( \mathbb{R}^4 \)?

No: Not closed under scalar mult:

Since \((1, 0, 0, 0)\) is in the set but \((1)(1, 0, 0, 0)\) is not in the set

(c) Is \( \{(a, b, a - b + 2, a + b) \mid a, b \text{ arbitrary} \} \) a subspace of \( \mathbb{R}^4 \)?

No - \((0, 0, 0, 0)\) is not in the set.
4. (10 pts) Show that the functions

\[ f_1(x) = e^{2x}, \quad f_2(x) = e^{-x} \]

are linearly independent on the interval \( I = (-\infty, \infty) \), by computing their Wronskian.

\[
\begin{vmatrix}
  f_1 & f_2 \\
  f'_1 & f'_2 \\
\end{vmatrix} = 
\begin{vmatrix}
  e^{2x} & e^{-x} \\
  2e^{2x} & -e^{-x} \\
\end{vmatrix} = -e^{2x} - 2e^{-x} = -3e^x
\]

Since this is non-zero at one or more points of \( I \), we must have linear independence.

(See Theorem 4.5.21)
5. (22 pts) In each part, you must provide a valid reason for your answers to receive full credit.

(a) Consider the subset \( S_1 = \{(1,2,1), (1,2,-1)\} \) of \( \mathbb{R}^3 \).

i. Is \( S_1 \) linearly independent? Why or why not?

Yes - clearly not multiples

ii. Does \( S_1 \) span \( \mathbb{R}^3 \)? Why or why not?

No - at least 3 vectors are needed to span \( \mathbb{R}^3 \) since \( \dim(\mathbb{R}^3) = 3 \).

iii. Is \( S_1 \) a basis for \( \mathbb{R}^3 \)? Why or why not?

No - since \( S_1 \) does not span \( \mathbb{R}^3 \).

(b) Consider the subset \( S_2 = \{(1,2,1), (1,0,-1), (1,6,5)\} \) of \( \mathbb{R}^3 \).

i. Is \( S_2 \) linearly independent? Why or why not?

No:

\[
\begin{vmatrix}
2 & 1 & 1 \\
1 & -1 & 5 \\
1 & 1 & 1
\end{vmatrix} = -2 \begin{vmatrix}
-1 & 1 \\
1 & 1
\end{vmatrix} - 6 \begin{vmatrix}
1 & 1 \\
1 & 1
\end{vmatrix} = 0.
\]

So lin. dependent.

Or just note \( 3(\frac{1}{2}) - 2(\frac{1}{2}) = (\frac{1}{2}) \) is not a linear combination of \( (1,2,1) \) and \( (1,0,-1) \).

ii. Does \( S_2 \) span \( \mathbb{R}^3 \)? Why or why not?

No: A set of 3 vectors spans \( \mathbb{R}^3 \) if and only if the set is lin. independent.

iii. Is \( S_2 \) a basis for \( \mathbb{R}^3 \)? Why or why not?

No - does not span nor is lin. indep.
(c) Consider the subset \( S_3 = \{(0, -3, 1), (1, 1, -1), (4, 1, -3), (1, 0, 0)\} \) of \( \mathbb{R}^3 \).

i. Is \( S_3 \) linearly independent? Why or why not?

\[ \text{No: Any set with 4 or more vectors in } \mathbb{R}^3 \text{ must be lin. dep. since } \dim(\mathbb{R}^3) = 3. \]

ii. Does \( S_3 \) span \( \mathbb{R}^3 \)? Why or why not?

\[ \text{Yes: Solution 1: Let } A = \begin{pmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & -3 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \]

\[ \text{Now reduce: } \begin{pmatrix} 1 & -1 & -3 & 1 \\ 0 & -2 & -8 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

We see columns 1, 2, 3, 4 of \( A \) are a basis for \( \text{colspace}(A) \) (see Theorem 4.8.6).

In particular, \( \text{colspace}(A) \) has dimension 3, so it must be all of \( \mathbb{R}^3 \).

So, the columns of \( A \) (which is \( S_3 \)) span \( \mathbb{R}^3 \). (See Next Page For solution 2 of this part)

iii. Is \( S_3 \) a basis for \( \mathbb{R}^3 \)? Why or why not?

\[ \text{No: not lin. indep.} \]

(d) Give TWO examples of a basis of \( \mathbb{R}^3 \)

Here's 5 examples:

\[ \text{Ex} 1: \sum (1, 0, 0), (0, 1, 0), (0, 0, 1) \]

\[ \text{Ex} 2: \sum (2, 0, 0), (0, 1, 0), (0, 0, 1) \]

\[ \text{Ex} 3: \sum (1, 1, 0), (1, -1, 0), (0, 0, 1) \]

\[ \text{Ex} 4: \sum (1, 2, 0), (3, 1, 0), (0, 0, 1) \]

\[ \text{Ex} 5: \sum (1, 1, 1), (1, -1, 0), (1, 0, 0) \]
Solution 2: We show that

\[ a(0, -3, 1) + b(1, 1, -1) + c(4, 1, -3) + d(1, 0, 0) = (y_1, y_2, y_3) \]

has a solution for any \((y_1, y_2, y_3) \in \mathbb{R}^3\).

\[ \Delta \rightarrow b + 4c + d = y_1 \]

\[ -3a + b + c = y_2 \]

\[ a - b - 3c = y_3 \]

\[ \rightarrow \begin{pmatrix} 0 & 1 & 4 & 1 & y_1 \\ -3 & 1 & 1 & 0 & y_2 \\ 1 & -1 & -3 & 0 & y_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -3 & 0 & y_3 \\ 0 & 1 & 4 & 1 & y_1 \\ -3 & 1 & 1 & 0 & y_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -3 & 0 & y_3 \\ 0 & 1 & 4 & 1 & y_1 \\ 0 & 0 & 0 & 2 & 2y_1 + y_2 + 3y_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -3 & 0 & y_3 \\ 0 & 1 & 4 & 1 & y_1 \\ 0 & 0 & 0 & 1 & y_1 + \frac{y_2}{3} + \frac{3}{2}y_3 \end{pmatrix} \]

Regardless of the values of \(y_1, y_2, y_3\), the system is consistent. Therefore, these 4 vectors span \(\mathbb{R}^3\).

Solution 3: Could also show 1st, 2nd, 4th vectors form a lin. indep. set of 3 vectors in \(\mathbb{R}^3\), hence they span \(\mathbb{R}^3\). Clearly it follows that \(S_3\) does as well.
6. (14 pts) Find a linearly independent subset of the following set of vectors that spans the same subspace of \( \mathbb{R}^4 \):

\[
\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -4 \\ -1 \end{pmatrix} \right\}
\]

Let \( A = \begin{pmatrix} -1 & 2 & -4 & 1 & 2 \\ 1 & -1 & 5 & -1 & 2 \\ 0 & 2 & 2 & -3 & -4 \\ 0 & -3 & -3 & 1 & -1 \end{pmatrix} \) so \( \text{colspan}(A) \) is the space spanned by the vectors.

We find a basis for \( \text{colspan}(A) \) using Thm 4.8.6 using this method it will be a subset of the original columns.

Row reduce:

\[
\begin{pmatrix} 1 & 2 & -4 & 1 & 2 \\ -1 & 1 & 5 & -1 & 2 \\ 0 & 2 & 2 & -3 & -4 \\ 0 & -3 & -3 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -4 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & -3 & -4 \\ 0 & -3 & -3 & 1 & -1 \end{pmatrix}
\]

We conclude that columns 1, 2, 4 of \( A \) are a basis for \( \text{colspan}(A) \). In particular, \( \mathcal{B}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -3 \\ -3 \end{pmatrix} \right\} \) is a linearly independent subset of the given set that spans the same subspace of \( \mathbb{R}^4 \).
7. (20 pts) Let \( A = \begin{pmatrix} 1 & -2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & -4 & 6 & 0 & 8 \\ 3 & -6 & 9 & -1 & 10 \end{pmatrix} \).

(a) The nullspace(\(A\)) is a subspace of \(\mathbb{R}^k\) for what value of \(k\)?

\[ \text{nullspace}(A) = \mathbb{R}^5 \]

(b) Find a basis for and the dimension of nullspace(\(A\)).

\[
\begin{pmatrix} 1 & -2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & -4 & 6 & 0 & 8 \\ 3 & -6 & 9 & -1 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[ \text{nullspace}(A) = \text{span} \left\{ \begin{pmatrix} 2r \\ -3s \\ 3r \\ -4t \\ s \end{pmatrix} \right\} \quad \text{where} \quad r, s, t \in \mathbb{R} \]

(c) The rowspace(\(A\)) is a subspace of \(\mathbb{R}^m\) for what value of \(m\)?

\[ \text{nullspace}(A) = \text{span} \left\{ \begin{pmatrix} 2r_1 \\ -3s_1 \\ 3r_1 \\ -4t_1 \\ s_1 \end{pmatrix} \right\} \quad \text{where} \quad r_1, s_1 \in \mathbb{R} \]

(d) Find a basis for and the dimension of rowspace(\(A\)).

\[ \text{nullspace}(A) = \text{span} \left\{ \begin{pmatrix} 2 \\ -3 \\ 3 \\ -4 \\ 1 \end{pmatrix} \right\} \]

\[ \text{dim}(\text{nullspace}(A)) = 3 \]

Since these 3 vectors are \( \text{lin. indep.} \), they are a basis for nullspace(\(A\)).

\[ \text{nullspace}(A) = \mathbb{R}^3 \]

(e) Therefore, \( \text{rowspace}(B) = \text{rowspace}(A) \) (see theorem 4.8.01).

Then, since \( \left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \\ 4 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\} \) is a basis for \text{rowspace}(B),

it is also a basis for \text{rowspace}(A).

\[ \text{dim(rowspace}(A) = 2 \]