1. (20 points)

What is the surface area of the surface of revolution obtained by rotating the infinite curve $e^{-x}, x \geq 0$ around the $x$-axis?

You may use the formula

$$\int \sec^3(x) \, dx = \frac{\sec(x) \tan(x) + \ln|\sec(x) + \tan(x)|}{2} + C.$$ 

Solution:

We have $y' = -e^{-x}$ so

$$A = 2\pi \int_{1}^{\infty} y \sqrt{1 + y'^2} \, dx$$

$$= 2\pi \int_{0}^{\infty} e^{-x} \sqrt{1 + e^{-2x}} \, dx$$

$$= -2\pi \int_{1}^{0} \sqrt{1 + u^2} \, du$$

where $u = e^{-x}$

$$= 2\pi \int_{0}^{\infty} \sqrt{1 + u^2} \, du$$

where $u = \tan \theta$

$$= 2\pi \int_{0}^{\pi/4} \sec^3 \theta d\theta$$

so $du = \sec^2 \theta d\theta$

and $\sqrt{1 + u^2} = \sec \theta$

$$= \frac{\pi}{2 \sqrt{2}}$$

$$= \frac{\pi}{\sqrt{2} + \ln(1 + \sqrt{2})}.$$
2. (20 points)

Consider the parametric curve (an astroid or 4 pointed hypocycloid) \( x = \cos^3(t), \ y = \sin^3(t), \ t \in [0, 2\pi]. \)

(a) (7 points)  At what points is the tangent horizontal or vertical?

(b) (6 points)  At what points does it have slope \( \pm 1? \)

(c) (7 points)  Find the equation of the form \( y = mx + b \) for the tangent at \( t = \frac{\pi}{4}. \)

Solution: (a) We have

\[
\frac{dx}{dt} = -3 \sin t \cos^2 t \\
\frac{dy}{dt} = 3 \cos t \sin^2 t \\
\frac{dy}{dx} = -\frac{\cos t \sin^2 t}{\sin t \cos^2 t} = -\tan t
\]

The tangent line is horizontal when this derivative is 0, namely when \( t = 0 \) and \( t = \pi. \) The tangent line is vertical when the derivative is undefined, namely at \( t = \frac{\pi}{2} \) and \( t = \frac{3\pi}{2}. \)

Solution: (b) The slope of the tangent line is \( \pm 1 \) when \( t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \) and \( \frac{7\pi}{4}. \)

Solution: (c) At \( t = \frac{\pi}{4} \) we have \( x = y = \sqrt{2}/4 \) and \( dy/dx = -1, \) so the equation for the tangent line is

\[
\frac{y - \sqrt{2}/4}{x - \sqrt{2}/4} = -1 \\
y - \sqrt{2}/4 = -(x - \sqrt{2}/4) \\
= -x + \sqrt{2}/4 \\
y = -x + \sqrt{2}/2.
\]
3. (20 points)

Find the arc length of the cycloid \( x = r(t - \sin(t)) \) and \( y = r(1 - \cos(t)) \), for \( 0 \leq t \leq 2\pi \).

Solution: We have

\[
\frac{dx}{dt} = r(1 - \cos t) \\
\frac{dy}{dt} = r \sin t
\]

\[
\left( \frac{ds}{dt} \right)^2 = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2
\]

\[
= r^2 ((1 - \cos t)^2 + \sin^2 t)
\]

\[
= r^2 (1 - 2 \cos t + \cos^2 t + \sin^2 t)
\]

\[
= 2r^2 (1 - \cos t)
\]

\[
\frac{ds}{dt} = 2r \sqrt{\frac{1 - \cos t}{2}}
\]

\[
= 2r \sin(t/2),
\]

so the arc length is

\[
s = \int_{0}^{2\pi} 2r \sin(t/2) dt
\]

\[
= 4r \int_{0}^{\pi} \sin u du \quad \text{where } u = t/2
\]

\[
= -4r \cos u \bigg|_{0}^{\pi}
\]

\[
= 8r
\]
4. (20 points)

Consider the logarithmic spiral \( r = e^\theta, \theta \geq 0 \), which can be defined parametrically by \( x = e^t \cos t \) and \( y = e^t \sin t \) with \( t = \theta \).

(a) (10 points) Calculate the arc-length of the logarithmic spiral for \( 0 \leq \theta \leq b \).

(b) (10 points) Calculate the area of the region between the \( x \)-axis and the curve for \( 0 \leq \theta \leq \pi \).

Solution: (a) For the arc length we have

\[
\frac{dx}{dt} = e^t (\cos t - \sin t) \\
\frac{dy}{dt} = e^t (\sin t + \cos t) \\
\left( \frac{ds}{dt} \right)^2 = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \\
= e^{2t} ((\cos t - \sin t)^2 + (\sin t + \cos t)^2) \\
= e^{2t} ((\cos^2 t - 2\cos t \sin t + \sin^2 t) + (\cos^2 t + 2\cos t \sin t + \sin^2 t)) \\
= 2e^{2t} \\
\frac{ds}{dt} = e^t \sqrt{2},
\]

so

\[
s = \sqrt{2} \int_0^b e^t dt = \sqrt{2} e^t \bigg|_0^b = \sqrt{2}(e^b - 1).
\]

Solution: (b) Using the area formula for polar coordinates, we have

\[
A = \frac{1}{2} \int_0^\pi r^2 d\theta \\
= \frac{1}{2} \int_0^\pi e^{2\theta} d\theta \\
= \frac{1}{2} e^{2\theta} \bigg|_0^\pi \\
= \frac{e^{2\pi}}{2} - \frac{1}{2} \\
= e^{2\pi} - 1.
\]
5. (20 points)

(a) (5 points) Use L’Hospital’s Rule to show that for $k \gt 0$,

$$
\lim_{x \to \infty} x^k e^{-x^2} = \frac{k}{2} \lim_{x \to \infty} x^{k-2} e^{-x^2}.
$$

(b) (5 points) Let $a_n = n^8 e^{-n^2}$ where $n = 1, 2, 3, \ldots$. Show that the sequence $\{a_n : n \geq 1\}$ converges. What is the limit?

(c) (5 points) Does the sequence $b_n = \cos\left(\frac{n\pi}{2}\right)(-\frac{1}{2})^n$ converge? Why or why not?

(d) (5 points) Does the sequence $b_n = \frac{1}{n^{0.005}}$ converge? Why or why not?

**Solution:** (a) We have

$$
\lim_{x \to \infty} x^k e^{-x^2} = \lim_{x \to \infty} x^k \frac{e^{-x^2}}{e^{x^2}}
$$

$$
= \lim_{x \to \infty} \frac{kx^{k-1}}{2x e^{x^2}}
$$

$$
= \frac{k}{2} \lim_{x \to \infty} x^{k-2} e^{-x^2}
$$

$$
= \frac{k}{2} \lim_{x \to \infty} x^{k-2} e^{-x^2},
$$

**Solution:** (b) From (a) we see that

$$
\lim_{x \to \infty} x^8 e^{-x^2} = 4 \lim_{x \to \infty} x^6 e^{-x^2} = 12 \lim_{x \to \infty} x^4 e^{-x^2} = 24 \lim_{x \to \infty} x^2 e^{-x^2} = 24 \lim_{x \to \infty} e^{-x^2} = 0,
$$

so the sequence converges to 0.

**Solution:** (c) Since $-1 \leq \cos\left(\frac{n\pi}{2}\right) \leq 1$, $-1/2^n \leq b_n \leq 1/2^n$, so $\lim_{n \to \infty} b_n = 0$.

**Solution:** (d) Since $\lim_{n \to \infty} n^{0.005} = \infty$, $\lim_{n \to \infty} b_n = 0$. 

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