1. (20 points)
(a) Use integration by parts to express $\int x^n e^x \, dx$ in terms of $\int x^{n-1} e^x \, dx$ for $n > 0$.

(b) Use the formula repeatedly to find
$$\int x^3 e^x \, dx.$$ 
You will not get partial credit here if the formula you are using is incorrect.

Solution: a.) Let $u = x^n$ and $dv = e^x \, dx$, so $du = n x^{n-1} \, dx$ and $v = e^x$. Then
$$\int x^n e^x \, dx = \int u \, dv = uv - \int v \, du = x^n e^x - n \int x^{n-1} e^x \, dx$$

b.) We have
$$\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx$$
$$= x^3 e^x - 3 \left( x^2 e^x - 2 \int x e^x \, dx \right)$$
$$= x^3 e^x - 3 \left( x^2 e^x - 2 (x e^x - \int e^x \, dx) \right)$$
$$= x^3 e^x - 3 \left( x^2 e^x - 2 (x e^x - e^x) \right) + c$$
$$= (x^3 - 3x^2 + 6x - 6) e^x + c.$$

2. (20 points) The half bagel problem. Consider the region between the $x$-axis and the semicircle $y = \sqrt{a^2 - (x - b)^2}$ with $b > a > 0$. The semicircle has radius $a$ and center $(b, 0)$. We want to find the volume $V$ of the solid of revolution about the $y$-axis.

(a) Write the integral for the volume and convert it to a trig integral using the substitution $x = b - a \cos \theta$ for $0 \leq \theta \leq \pi$. 
(b) Find the volume in terms of \(a\) and \(b\).

*You will not get partial credit here if the integral you are using is incorrect.*

**Solution:** (a) Since \(x = b - a \cos \theta\), we have \(dx = a \sin \theta \, d\theta\) and

\[
y = \sqrt{a^2 - (x - b)^2} = \sqrt{a^2 - (-a \cos \theta)^2} = a \sqrt{1 - \cos^2 \theta} = a \sin \theta.
\]

The integral for the volume is

\[
V = \int_{b-a}^{b+a} 2\pi xy \, dx = 2\pi \int_0^\pi (b - a \cos \theta)(a \sin \theta) a \sin \theta \, d\theta
\]

(b) Our integral is

\[
V = 2\pi \int_0^\pi (b - a \cos \theta)(a \sin \theta) a \sin \theta \, d\theta
\]

\[
= 2\pi a^2 b \int_0^\pi \sin^2 \theta \, d\theta - 2\pi a^3 \int_0^\pi \sin^2 \theta \cos \theta \, d\theta
\]

\[
= 2\pi a^2 b \int_0^\pi \frac{1 - \cos 2\theta}{2} \, d\theta - 2\pi a^3 \int_0^\pi u^2 \, du \quad \text{where} \ u = \sin \theta
\]

\[
= 2\pi a^2 b \int_0^{2\pi} \frac{1 - \cos w}{4} \, dw \quad \text{where} \ w = 2\theta
\]

\[
= \frac{\pi a^2 b}{2} (w - \sin w)|_{0}^{2\pi} \, dw
\]

\[
= \pi^2 a^2 b.
\]

3. (20 points)

(a) Let \(a > 0\) be a fixed positive number. Compute the definite integral

\[
\int_0^{a/\sqrt{2}} \frac{1}{(a^2 - x^2)^{3/2}} \, dx.
\]

Your answer should be expressed in terms of \(a\).

(b) Find the integral

\[
\int \frac{1}{\sqrt{x^2 + 2x + 10}} \, dx.
\]

**Solution:** (a)
We set \( x = a \sin \theta \). Then \( dx = a \cos \theta d\theta \) and \( \sqrt{a^2 - x^2} = a \cos \theta \). Also when \( x = 0 \), \( \sin \theta = 0 \) so that \( \theta = 0 \), and when \( x = \frac{a}{\sqrt{2}} \), \( \sin \theta = \frac{1}{\sqrt{2}} \) so that \( \theta = \frac{\pi}{4} \). The definite integral becomes

\[
\int_0^{a/\sqrt{2}} \frac{1}{(a^2 - x^2)^{3/2}} \, dx = \int_0^{\pi/4} \frac{a \cos \theta}{(a \cos \theta)^3} \, d\theta = \frac{1}{a^2} \int_0^{\pi/4} \sec^2 \theta \, d\theta = \frac{1}{a^2} \tan \theta \bigg|_0^{\pi/4} = \frac{1}{a^2}.
\]

(b) We complete the square \( x^2 + 2x + 10 = (x + 1)^2 + 9 \). Then consider the substitution \( u = x + 1 \), so that \( du = dx \), and we find

\[
\int \frac{1}{\sqrt{x^2 + 2x + 10}} \, dx = \int \frac{1}{\sqrt{(x + 1)^2 + 9}} \, dx = \int \frac{1}{\sqrt{u^2 + 9}} \, du.
\]

Next we use a trig substitution. Let \( u = 3 \tan \theta \). Then \( du = 3 \sec^2 \theta \, d\theta \) and \( \sqrt{u^2 + 9} = 3 \sec \theta \), so that

\[
\int \frac{1}{\sqrt{x^2 + 2x + 10}} \, dx = \int \frac{1}{\sqrt{u^2 + 9}} \, du = \int \frac{1}{3 \sec \theta} \, 3 \sec^2 \theta \, d\theta = \int \sec \theta \, d\theta
\]

\[
= \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{\sqrt{u^2 + 9}}{3} + \frac{u}{3} \right| + C
\]

\[
= \ln \left| \frac{(x + 1)^2 + 9}{3} + \frac{x + 1}{3} \right| + C
\]

\[
= \ln \left| \sqrt{(x + 1)^2 + 9} + x + 1 \right| + C
\]

4. (20 points)

(a) Evaluate the integral

\[
\int \frac{2x^3 + 5x^2 + x}{x^3 + x^2 - x - 1} \, dx.
\]

(b) Find the integral

\[
\int \sin^5 x \cos^2 x \, dx.
\]

Solution: (a)

Note that \( 2x^3 + 5x^2 + x = (x^3 + x^2 - x - 1) \cdot 2 + (3x^2 + 3x + 2) \). So,

\[
\frac{2x^3 + 5x^2 + x}{x^3 + x^2 - x - 1} = 2 + \frac{3x^2 + 3x + 2}{x^3 + x^2 - x - 1}.
\]
Factoring the denominator,
\[ x^3 + x^2 - x - 1 = (x + 1)^2(x - 1). \]

So, we may write
\[ \frac{3x^2 + 3x + 2}{x^3 + x^2 - x - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}. \]

Summing the terms on the right hand side and comparing the numerators on both sides, we get
\[ 3x^2 + 3x + 2 = A(x + 1)^2 + B(x - 1)(x + 1) + C(x - 1). \]

Plugging \( x = 1 \) in the above equation, \( 8 = 4A \). So, \( A = 2 \).

Plugging \( x = -1 \) in the above equation, \( 2 = -2C \). So, \( C = -1 \).

Comparing the coefficients of \( x^2 \), \( 3 = A + B \). So, \( B = 1 \).

So,
\[
\int \frac{2x^3 + 5x^2 + x}{x^3 + x^2 - x - 1} \, dx = \int \left( 2 + \frac{2}{x - 1} + \frac{1}{(x + 1)} - \frac{1}{(x + 1)^2} \right) \, dx
\]
\[
= 2x + 2 \ln |x - 1| + \ln |x + 1| + \frac{1}{x + 1} + C.
\]

(b)

Note that \( \sin^5 x = \sin^4 x \cdot \sin x = (1 - \cos^2 x)^2 \sin x \). So, letting \( u = \cos x \), we have \( du = -\sin x \, dx \) and
\[
\int \sin^5 x \cos^2 x \, dx = \int (1 - \cos^2 x)^2 \sin x \cos^2 x \, dx
\]
\[
= - \int (1 - u^2)^2 u^2 \, du
\]
\[
= - \int u^6 - 2u^4 + u^2 \, du
\]
\[
= - \left( \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} \right) + C
\]
\[
= -\frac{1}{7} \cos^7 x + \frac{2}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C.
\]
5. (20 points)

A spring is attached to a wall. In its resting position, the end of the spring is 1 m away from the wall. It takes 16 J of work to pull the spring so that the end is 3 m away from the wall. If the spring is brought back to rest, how much work does it then take to pull its end to 6 m away from the wall?

Solution: Let $x$ be the distance that the spring is stretched from the resting position, and let $k$ be the spring constant. By Hooke’s law we have $F = kx$. So

$$\text{Work} = \int_0^{3-1} kx \, dx = \left[ \frac{kx^2}{2} \right]_0^2 = \frac{k(2)^2}{2} = 2k \, J$$

and the work is 16 J, so $k = 8$. To pull the end to 6 m away from the wall, this is 5 m from rest, so it takes

$$\text{Work} = \int_0^5 8x \, dx = 4x^2 \bigg|_0^5 = 4(5)^2 = 4(25) = 100 \, J$$