1. (20 points)

A swimming pool is 20 feet wide, 50 feet long and 5 feet deep. It is partly filled with water to a depth of 4 feet. How much work is needed to pump all the water out of the pool by lifting it to the top of the pool, which is five feet above the bottom? Your answer should be expressed in foot-pounds. Assume that the density of water is 60 pounds per cubic foot.

Solution: Let \( x \) be the distance in feet to the top of the pool, so \( 1 \leq x \leq 5 \). For each horizontal layer of water we have

\[
\begin{align*}
\text{area} & = 20 \text{ feet} \times 50 \text{ feet} = 1000 \text{ square feet} \\
\text{volume} & = 1000 \, dx \text{ cubic feet} \\
\text{weight} & = 60,000 \, dx \text{ pounds} \\
\text{work need to lift} & = 60,000x \, dx \text{ foot-pounds,}
\end{align*}
\]

so the total amount of work required is

\[
\int_{1}^{5} 60,000x \, dx = 60,000 \int_{1}^{5} x \, dx = 60,000 \left. \frac{x^2}{2} \right|_{1}^{5} = 30,000(25 - 1) = 720,000 \text{ foot-pounds.}
\]

Alternatively, let \( x \) be the distance from the bottom of the pool, so \( 0 \leq x \leq 4 \). Then the work needed to lift a horizontal layer of water is \( 60,000(5 - x) \, dx \) foot-pounds, so the total work required is

\[
\int_{0}^{4} 60,000(5 - x) \, dx = 60,000 \int_{0}^{4} (5 - x) \, dx = 60,000 \left. \left(5x - \frac{x^2}{2}\right)\right|_{0}^{4} = 60,000(20 - 8) = 720,000 \text{ foot-pounds.}
\]

2. (20 points)
(a) Find the integral
\[ \int \frac{4x^2}{(x^2 + 1)(x^2 - 1)} \, dx. \]

(b) Find the integral
\[ \int sec^3 x \tan^3 x \, dx. \]

**Solution: (a)**

First, find \( A, B, C \) and \( D \) such that
\[
\frac{4x^2}{(x^2 + 1)(x^2 - 1)} = \frac{4x^2}{(x^2 + 1)(x - 1)(x + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{x + 1}.
\]

To find \( C \), multiply \( x - 1 \) both sides and obtain
\[
\frac{4x^2}{(x^2 + 1)(x + 1)} = \frac{Ax + B}{x^2 + 1} (x - 1) + C + \frac{D}{x + 1} (x - 1).
\]

Then by letting \( x = 1 \) we get \( C = 1 \). Similarly by multiplying \( x + 1 \) and then letting \( x = -1 \), we obtain \( D = -1 \). To find \( A \) and \( B \), observe
\[
\frac{4x^2}{(x^2 + 1)(x^2 - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{1}{x - 1} + \frac{-1}{x + 1}
\]
\[
= \frac{Ax + B}{x^2 + 1} + \frac{2}{x^2 - 1}
\]
\[
= \frac{(Ax + B)(x^2 - 1) + 2(x^2 + 1)}{(x^2 + 1)(x^2 - 1)}
\]

Comparing coefficients of numerators both sides, we have \( A = 0 \) and \( B = 2 \). Hence,
\[
\frac{4x^2}{(x^2 + 1)(x^2 - 1)} = \frac{2}{x^2 + 1} + \frac{1}{x - 1} + \frac{-1}{x + 1}
\]

and
\[
\int \frac{4x^2}{(x^2 + 1)(x^2 - 1)} \, dx = 2 \arctan x + \ln |x - 1| - \ln |x + 1| + C.
\]

Note: We may do this by observing
\[
\frac{4x^2}{(x^2 + 1)(x^2 - 1)} = \frac{2}{x^2 - 1} + \frac{2}{x^2 + 1}
\]

and
\[
\frac{2}{x^2 - 1} = \frac{1}{x - 1} - \frac{1}{x + 1}.
\]
(b)

Let \( u = \sec x \), then \( du = \sec x \tan x \, dx \). Since \( \tan^2 x = \sec^2 x - 1 = u^2 - 1 \), we obtain

\[
\int \sec^3 x \tan^3 x \, dx = \int u^2(u^2 - 1) \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.
\]

3. (20 points)

(a) Find the integral

\[
\int_0^\sqrt{3} \frac{dx}{(9 + x^2)^{3/2}}.
\]

(b) Find the integral

\[
\int \frac{\cos x}{\sqrt{2 \sin x + 1 - \cos^2 x}} \, dx.
\]

**Solution: (a)**

Let \( x = 3 \tan \theta \). Then \( dx = 3 \sec^2 \theta \, d\theta \), and \( 9 + x^2 = 9 + 9 \tan^2 \theta = 9 \sec^2 \theta \). Also, when \( x = 0 \) we have \( \tan \theta = 0 \), so that \( \theta = 0 \) (recall \(-\pi/2 < \theta < \pi/2\)), and when \( x = \sqrt{3} \) we have \( \sqrt{3} = 3 \tan \theta \) so that \( \tan \theta = \frac{1}{\sqrt{3}} \), and \( \theta = \pi/6 \). We find

\[
\int_0^\sqrt{3} \frac{dx}{(9 + x^2)^{3/2}} = \int_0^{\pi/6} \frac{3 \sec^2 \theta \, d\theta}{(9 \sec^2 \theta)^{3/2}} = \frac{1}{9} \int_0^{\pi/6} \frac{d\theta}{\sec \theta} = \frac{1}{9} \int_0^{\pi/6} \cos \theta \, d\theta = \frac{\sin \theta}{9} \bigg|_0^{\pi/6} = \frac{1}{18}.
\]

(b)

Let \( u = \sin x \). Then \( du = \cos x \, dx \), and with \( 1 - \cos^2 x = \sin^2 x \) we have

\[
\int \frac{\cos x}{\sqrt{2 \sin x + 1 - \cos^2 x}} \, dx = \int \frac{du}{\sqrt{2u + u^2}} = \int \frac{du}{\sqrt{(u + 1)^2 - 1}}.
\]

Next we set \( u + 1 = \sec \theta \). Then \( du = \sec \theta \tan \theta \, d\theta \). Also \((u + 1)^2 - 1 = \sec^2 \theta - 1 = \tan^2 \theta\).
Then

\[
\int \frac{du}{\sqrt{(u+1)^2-1}} = \int \frac{\sec \theta \tan \theta \, d\theta}{\sqrt{\tan^2 \theta}} = \int \sec \theta \, d\theta
\]

\[= \ln |\sec \theta + \tan \theta| + C\]

\[= \ln |u + 1 + \sqrt{(u+1)^2-1}| + C\]

\[= \ln |\sin x + 1 + \sqrt{\sin x + \cos^2 x + 2 \sin x}| + C\]

\[= \ln |\sin x + 1 + \sqrt{\sin^2 x + 2 \sin x}| + C.\]

4. (20 points)

(a) Use integration by parts to prove the reduction formula

\[
\int \cos^n x \, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx \quad \text{for } n \geq 2.
\]

(b) Use the formula to find

\[
\int_0^{\pi/2} \cos^3 x \, dx.
\]

Solution:

a.) Let \( u = \cos^{n-1} x \), so \( du = (n-1) \cos^{n-2} x (-\sin x) \, dx \). Let \( dv = \cos x \, dx \), so \( v = \sin x \).

Then:

\[
\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx
\]

\[= \int u \, dv = uv - \int v \, du
\]

\[= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx
\]

\[= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x - \cos^n x \, dx
\]

\[= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx
\]

\[n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx
\]

Now divide by \( n \).
b.) Let \( n = 3 \).

\[
\int \cos^3 x \, dx = \frac{\sin x \cos^3 x}{3} + \frac{3 - 1}{3} \int \cos^3 x \, dx
\]

\[
= \frac{\sin x \cos^2 x}{3} + \frac{2}{3} \sin x + C
\]

So

\[
\int_0^{\pi/2} \cos^3 x \, dx = \frac{\sin x \cos^2 x}{3} + \frac{2}{3} \sin x \bigg|_0^{\pi/2} = \frac{2}{3}
\]

5. (20 points) Consider the region bounded by the \( x \)-axis and the curve \( y = \sin x \) for \( 0 \leq x \leq \pi \).

(a) Find the volume of the solid obtained by rotating it about the \( x \)-axis.

(b) Find the volume of the solid obtained by rotating the same region about the \( y \)-axis.

Solution:

a) Integrating with respect to \( x \) makes this a washer method problem with

\[
V = \int_0^{\pi} \pi \sin^2 x \, dx
\]

\[
= \pi \int_0^{\pi} \left( 1 - \cos 2x \right) \, dx \quad \text{by the half angle formula}
\]

\[
= \pi \left( \int_0^{\pi} \frac{dx}{2} - \int_0^{2\pi} \frac{\cos u}{4} \, du \right) \quad \text{where } u = 2x
\]

\[
= \left( \frac{\pi x}{2} \right) \bigg|_0^{\pi} - \left( \frac{\sin u}{4} \right) \bigg|_0^{2\pi}
\]

\[
= \frac{\pi^2}{2}
\]

b) Integrating with respect to \( x \) makes this a shell method problem with

\[
V = \int_0^{\pi} 2\pi x \sin x \, dx = 2\pi \int_0^{\pi} x \sin x \, dx
\]

For this we need integration by parts with

\[
u = x \quad dv = \sin x \, dx
\]

\[
\frac{du}{dx} = dx \quad v = -\cos x
\]
Then we have

\begin{align*}
V &= 2\pi \int_{x=0}^{x=\pi} u \, dv \\
&= 2\pi \left( uv \bigg|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} v \, du \right) \\
&= 2\pi \left( -x \cos x \bigg|_{0}^{\pi} + \int_{0}^{\pi} \cos x \, dx \right) \\
&= 2\pi \left( -x \cos x + \sin x \bigg|_{0}^{\pi} \right) \\
&= 2\pi(\pi - 0) = 2\pi^2.
\end{align*}