1. (16 points) Consider the functions \( y = x^2 \) and \( y = 3x \).

(a) Sketch the region enclosed by the graphs of the given functions, and find the area of this region.

(b) Let \( S \) be the solid obtained by rotating the above region about the \( x \)-axis. Sketch \( S \), along with a typical cross-section of \( S \), and find the volume of \( S \) using the washer method (also called the cross-sectional method.)

Solution: (a) A sketch of a similar region is on page 426 in the textbook (page 448 in the 5th edition). The area of the region is

\[
A = \int_0^3 (3x - x^2) \, dx = \left[ \frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{3 \cdot 3^2}{2} - \frac{3^3}{3} = \frac{27}{2} = \frac{9}{2}.
\]

(b) A sketch of a similar solid of revolution is on page 426 in the textbook (448 in the 5th edition). Using washers, the volume is

\[
V = \int_0^3 (\pi (3x)^2 - \pi (x^2)^2) \, dx = \pi \left[ \frac{9x^3}{3} - \frac{x^5}{5} \right]_0^3 = \pi \left( \frac{9 \cdot 3^3}{3} - \frac{3^5}{5} \right) = \frac{162\pi}{5}.
\]
2. (16 points) Again consider the functions $y = x^2$ and $y = 3x$.

(a) Let $S$ be the solid obtained by rotating the region bounded by the graphs of these functions about the $y$-axis. Sketch $S$, along with a typical cylindrical shell inside $S$, and find the volume of $S$ using the cylindrical shells method.

(b) Let $S$ be the solid obtained by rotating the region bounded by the graphs of these functions about the line $x = -3$. Sketch $S$ and find the volume of $S$ using whichever method you want (washer method or cylindrical shells.)

Solution: (a) A similar problem is done in complete detail in your textbook on page 435 (page 457 of the 5th edition), so we only give the answer here.

\[
V = \int_0^3 (2\pi x) (3x - x^2) \, dx = 2\pi \int_0^3 (3x^2 - x^3) \, dx \\
= 2\pi \left[ x^3 - \frac{x^4}{4} \right]_0^3 = 2\pi \left( 27 - \frac{81}{4} \right) = \frac{27\pi}{2}
\]

(b) A similar problem is done using washers in your textbook on page 429 (pages 449-450 in the 5th edition), so we only give the answer here.

\[
V = \int_0^9 \left[ \pi(3 + \sqrt{y})^2 - \pi \left( 3 + \frac{y}{3} \right)^2 \right] \, dy \\
= \pi \int_0^9 \left( 9 + 6\sqrt{y} + y - 9 - 2y - \frac{y^2}{9} \right) \, dy \\
= \pi \int_0^9 \left( 6\sqrt{y} - y - \frac{y^2}{9} \right) \, dy \\
= \pi \left[ 4y^{3/2} - \frac{y^2}{2} - \frac{y^3}{27} \right]_0^9 = \pi \left( 108 - \frac{81}{2} - 27 \right) = \frac{81\pi}{2}.
\]

Using cylindrical shells, the radius is $(3 + x)$, the height is $(3x - x^2)$, and the volume is

\[
V = \int_0^3 2\pi (3 + x) (3x - x^2) \, dx = 2\pi \int_0^3 (9x - x^3) \, dx \\
= 2\pi \left[ \frac{9x^2}{2} - \frac{x^4}{4} \right]_0^3 = 2\pi \left( \frac{81}{2} - \frac{81}{4} \right) = \frac{81\pi}{2}.
\]
3. (10 points) A rectangular swimming pool is 10 meters long and 4 meters wide, the sides are 2 meters high and the depth of the water is 1.5 meters. How much work is required to pump out all the water over the side? (Note: Use $g = 9.8 m/s^2$ as the acceleration due to gravity and $1000 \text{ kg/m}^3$ as the density of water. Remember that 1 Joule = $1 \text{ kg m}^2\text{s}^{-2}$.)

**Solution:** Let $y_i^*$ be the height of the $i$th layer of water, $0 \leq y_i^* \leq 1.5$. An approximation to volume of the $i$th layer of water is

$$V_i \approx \text{(area)(thickness)} = 40 \Delta y.$$

The $i$th layer of water must travel a vertical distance of $2 - y_i^*$ to the top of the tank. Thus, the amount of work done pumping the water out of the tank is

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} (\text{density})(\text{volume})(\text{acceleration})(\text{distance})$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} (1000)(40 \Delta y)(9.8)(2 - y_i^*)$$

$$= 392000 \int_{0}^{1.5} (2 - y) \, dy$$

$$= 392000 \left[ 2y - \frac{y^2}{2} \right]_{0}^{1.5}$$

$$= 392000 \left( 3 - \frac{9}{8} \right)$$

$$= 392000 \frac{15}{8}$$

$$= 735000 \text{ Joules}.$$
4. (15 points) Evaluate the following integrals:

(a) \[ \int x^2 \cos(x^3 + 26) \, dx \]

(b) \[ \int_e^{2e} \frac{1}{x(\ln x)^3} \, dx \]

(c) \[ \int x^5 \sqrt{1 + x^2} \, dx \]

Solution: (a) Let \( u = x^3 + 26 \), then \( du = 3x^2 \, dx \) and

\[
\int x^2 \cos(x^3 + 26) \, dx = \frac{1}{3} \int \cos(u) \, du = \frac{1}{3} \sin(u) + C = \frac{\sin(x^3 + 26)}{3} + C
\]

Solution: (b) Let \( u = \ln x \), then \( du = \frac{1}{x} \, dx \) and when \( x = 2e \), \( u = \ln(2e) = 1 + \ln(2) \), when \( x = e \), \( u = 1 \), and

\[
\int_e^{2e} \frac{1}{x(\ln x)^3} \, dx = \left[ \frac{1}{u^2} \right]_1^{1+\ln(2)} = -\frac{1}{2u^2} \bigg|_1^{1+\ln(2)} = -\frac{1}{2(1 + \ln(2))^2} - \left( -\frac{1}{2} \right) = \frac{1}{2} - \frac{1}{2(1 + \ln(2))^2}
\]
Solution: (c) Let \( u = 1 + x^2 \), then \( du = 2x \, dx \), \( x^2 = u - 1 \), and
\[
\int x^5 \sqrt{1+x^2} \, dx = \int x^4 \sqrt{1+x^2} \, x \, dx
\]
\[= \frac{1}{2} \int (u - 1)^2 \sqrt{u} \, du \]
\[= \frac{1}{2} \int (u^2 - 2u + 1)u^{1/2} \, du \]
\[= \frac{1}{2} \int u^{5/2} - 2u^{3/2} + u^{1/2} \, du \]
\[= \frac{1}{2} \left[ \frac{2}{7} u^{7/2} - \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right] + C \]
\[= \frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} + \frac{u^{3/2}}{3} + C \]
\[= \frac{(1 + x^2)^{7/2}}{7} - \frac{2(1 + x^2)^{5/2}}{5} + \frac{(1 + x^2)^{3/2}}{3} + C \]
\[= \frac{(1 + x^2)^{7/2}}{105} (8 - 12x^2 + 15x^4) + C \]
5. (15 points) Evaluate the following integrals:

(a) \( \int x^2 e^x \, dx \)

(b) \( \int x \sin x \, dx \)

(c) \( \int \arctan(2x) \, dx \)

Solution: (a) Use integration by parts with \( u = x^2 \) and \( dv = e^x \, dx \), then \( du = 2x \, dx \) and \( v = e^x \) and

\[
\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx
\]

Using integration by parts a second time with \( u = x \) and \( dv = e^x \, dx \), then \( du = dx \) and \( v = e^x \), the integral becomes:

\[
x^2 e^x - 2 \left( xe^x - \int e^x \, dx \right) = x^2 e^x - 2[xe^x - e^x] + C = (x^2 - 2x + 2)e^x + C
\]

Solution: (b) Use integration by parts with \( u = x \) and \( dv = \sin x \, dx \), then \( du = dx \) and \( v = -\cos x \) and

\[
\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C
\]

Solution: (c) Use integration by parts with \( u = \arctan(2x) \) and \( dv = dx \), then \( du = \frac{2}{1 + (2x)^2} \, dx \) and \( v = x \) and

\[
\int \arctan(2x) \, dx = x \arctan(2x) - 2 \int \frac{x}{1 + (2x)^2} \, dx
\]

Now using the substitution \( u = 4x^2 \), then \( du = 8x \, dx \) and the integral becomes:

\[
x \arctan(2x) - \frac{1}{4} \int \frac{1}{1 + u} \, du = x \arctan(2x) - \frac{1}{4} \ln|1 + u| + C
\]

\[
= x \arctan(2x) - \frac{1}{4} \ln|1 + 4x^2| + C
\]
6. (14 points) Evaluate the following integrals:

(a) \[ \int_{0}^{10} \frac{1}{\sqrt{x - 10}} \, dx \]

(b) \[ \int \sin^5 \theta \cos^{10} \theta \, d\theta \]

Solution: (a) This is an improper integral:

\[ \int_{0}^{10} \frac{1}{\sqrt{x - 10}} \, dx = \lim_{b \to 10^-} \int_{0}^{b} \frac{1}{\sqrt{x - 10}} \, dx \]

Now, let \( u = x - 10 \), then \( du = dx \) and when \( x = 0, u = -10 \), when \( x = b, u = b - 10 \). Now the integral becomes:

\[ \lim_{b \to 10^-} \int_{-10}^{b-10} u^{-1/3} \, du = \lim_{b \to 10^-} \frac{3}{2} u^{2/3} \bigg|_{-10}^{b-10} = \frac{3}{2} [ (b - 10)^{2/3} - (-10)^{2/3} ] = -\frac{3}{2} 10^{2/3} \]

Solution: (b) Notice that the power of \( \sin \) is odd, so we factor out \( \sin \theta \) and write everything in terms of \( \cos \theta \):

\[ \int \sin^5 \theta \cos^{10} \theta \, d\theta = \int \sin^4 \theta \cos^{10} \theta \sin \theta \, d\theta = \int (\sin^2 \theta)^2 \cos^{10} \theta \sin \theta \, d\theta = \int (1 - \cos^2 \theta)^2 \cos^{10} \theta \sin \theta \, d\theta \]

Now let \( u = \cos \theta \), then \( du = -\sin \theta \, d\theta \) and

\[ \int \sin^5 \theta \cos^{10} \theta \, d\theta = -\int (1 - u^2)^2 u^{10} \, du = -\int (1 - 2u^2 + u^4) u^{10} \, du = -\int (u^{10} - 2u^{12} + u^{14}) \, du = -\left[ \frac{u^{11}}{11} - \frac{2u^{13}}{13} + \frac{u^{15}}{15} \right] + C = -\left[ \frac{\cos^{11} \theta}{11} - \frac{2\cos^{13} \theta}{13} + \frac{\cos^{15} \theta}{15} \right] + C = -\frac{\cos^{11} \theta}{11} + \frac{2\cos^{13} \theta}{13} - \frac{\cos^{15} \theta}{15} + C \]
7. (14 points) Evaluate the following integrals.

(a) \[ \int \frac{1}{\sqrt{49 + x^2}} \, dx \]

(b) \[ \int \frac{1}{x^2 + 8x + 15} \, dx \]

Solution: (a) Use the trigonometric substitution \( x = 7 \tan \theta \), then \( d\theta = 7 \sec^2 \theta \), and

\[
\int \frac{1}{\sqrt{49 + x^2}} \, dx = \int \frac{1}{\sqrt{49 + 49 \tan^2 \theta}} \, 7 \sec^2 \theta \, d\theta
\]

\[
= \int \frac{1}{\sqrt{49(1 + \tan^2 \theta)}} \, 7 \sec^2 \theta \, d\theta
\]

\[
= \int \frac{1}{\sec \theta} \, \sec^2 \theta \, d\theta
\]

\[
= \int \sec \theta \, d\theta
\]

\[
= \int \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} \, d\theta
\]

Now let \( u = \sec \theta + \tan \theta \), then \( du = \sec^2 \theta + \sec \theta \tan \theta \) and

\[
\int \frac{1}{\sqrt{49 + x^2}} \, dx = \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} \, d\theta
\]

\[
= \int \frac{1}{u} \, du
\]

\[
= \ln |u| + C
\]

\[
= \ln |\sec \theta + \tan \theta| + C
\]

And since \( x = 7 \tan \theta \), then \( \tan \theta = \frac{x}{7} \) and using a right triangle (or \( \tan^2 \theta + 1 = \sec^2 \theta \)) we get \( \sec \theta = \frac{\sqrt{x^2 + 49}}{7} \). Now:

\[
\int \frac{1}{\sqrt{49 + x^2}} \, dx = \ln |\sec \theta + \tan \theta| + C
\]

\[
= \ln \left| \frac{\sqrt{x^2 + 49}}{7} + \frac{x}{7} \right| + C
\]
Solution: (b) Factorizing the denominator we can re-write the integral

\[ \int \frac{1}{x^2 + 8x + 15} \, dx = \int \frac{1}{(x + 3)(x + 5)} \, dx \]

Now using partial fractions:

\[ \frac{1}{(x + 3)(x + 5)} = \frac{A}{x + 3} + \frac{B}{x + 5} = \frac{A(x + 5) + B(x + 3)}{(x + 3)(x + 5)} \]

Since the denominators are the same we can find \( A \) and \( B \) by setting the numerators equal to each other:

\[ 1 = A(x + 5) + B(x + 3) \]

This equation holds for all real numbers in particular for \( x = -5 \) we get \( B = -\frac{1}{2} \) and for \( x = -3 \) we get \( A = \frac{1}{2} \). Now the integral becomes:

\[ \int \frac{1}{x^2 + 8x + 15} \, dx = \int \frac{1}{2(x + 3)} - \frac{1}{2(x + 5)} \, dx \]

\[ = \frac{1}{2} \int \frac{1}{x + 3} \, dx - \frac{1}{2} \int \frac{1}{x + 5} \, dx \]

\[ = \frac{1}{2} \ln|x + 3| - \frac{1}{2} \ln|x + 5| + C \]