Part A
1. (20 points) Evaluate the integral
   \[ \int \frac{x^2}{\sqrt{4-x^2}} \, dx. \]

   **Solution:** (a) The simplest approach is to set \( x = 2 \sin \theta \). Then \( dx = 2 \cos \theta \, d\theta \) and
   \[ \sqrt{4-x^2} = \sqrt{4-4\sin^2 \theta} = \sqrt{4\cos^2 \theta} = 2 \cos \theta. \]
   So
   \[ \int \frac{x^2}{\sqrt{4-x^2}} \, dx = \int \frac{4\sin^2 \theta}{2 \cos \theta} \, 2 \cos \theta \, d\theta = \int 2 \sin^2 \theta \, d\theta = \int 2 \left(1 - \cos(2\theta)\right) \, d\theta = 2\theta - \sin(2\theta) + C \]
   To convert back to \( x \), we use the fact that \( \sin(2\theta) = 2 \sin \theta \cos \theta = \frac{1}{2} x \sqrt{4-x^2} \). So the answer becomes
   \[ 2 \arcsin \left(\frac{x}{2}\right) - \frac{1}{2} x \sqrt{4-x^2} + C. \]

2. (15 points)

   Find the volume of the solid obtained by rotating the region bounded by the curves \( y = \sqrt{x} \), \( x = 0 \), and \( y = 1 \) about the line \( y = 2 \).

   **Solution:** We use the washer method. First observe that (since \( y = \sqrt{x} \) intersects \( x = 0 \) and \( y = 1 \) at the points (0,0) and (0,1), respectively) the bounds on \( x \) this region are given by \( 0 \leq x \leq 1 \). Then for each \( x \) such that \( 0 \leq x \leq 1 \), the cross section of this solid with the plane passing through \( x \) perpendicular to the \( x \)-axis is an annulus (“washer”) of interior radius \( r_1(x) = 2 - 1 = 1 \) and exterior radius \( r_2(x) = 2 - \sqrt{x} \). This washer has area
A(x) = π(r_2(x))^2 - π(r_1(x))^2 = π((2 - \sqrt{x})^2 - 1). Therefore this solid has volume given by

\[
V = \int_0^1 A(x) \, dx = \pi \int_0^1 ((2 - \sqrt{x})^2 - 1) \, dx \\
= \pi \int_0^1 (3 - 4\sqrt{x} + x) \, dx \\
= \pi \left[ 3x - \frac{8x^{3/2}}{3} + \frac{x^2}{2} \right]_0^1 \\
= \pi \left( 3 - \frac{8}{3} + \frac{1}{2} \right) = \frac{5\pi}{6}.
\]

3. (15 points) Evaluate the integral

\[
\int \sin(x) \cos(x) e^{\sin x} \, dx.
\]

Solution: We use the substitution \( y = \sin x \). Then \( dy = \cos x \, dx \) and the integral becomes

\[
\int \sin x \cos x e^{\sin x} \, dx = \int ye^y \, dy.
\]

Next we integrate by parts. We let \( u = y \) and \( dv = e^y \, dy \), so that \( du = dy \) and \( v = e^y \). The integral becomes

\[
\int ye^y \, dy = ye^y - \int e^y \, dy = ye^y - e^y + C.
\]

Changing back to the variable \( x \) we conclude that

\[
\int \sin x \cos x e^{\sin x} \, dx = \sin xe^{\sin x} - e^{\sin x} + C.
\]

4. (20 points)

(a) Find the partial fraction expansion of

\[
\frac{1}{x^3 - 4x^2 + 4x}
\]

(b) Evaluate the integral

\[
\int \frac{1}{x^3 - 4x^2 + 4x} \, dx.
\]

(If your answer for part (a) is wrong, you will not receive credit for evaluating the integral of an incorrect function.)
Solution: (a) First we factor \( x^3 - 4x^2 + 4x \) as \( x(x-2)^2 \). So

\[
\frac{1}{x^3 - 4x^2 + 4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}
\]

for some constants \( A, B, \) and \( C \). Multiplying through by the denominator gives

\[
1 = A(x-2)^2 + Bx(x-2) + Cx.
\]

Setting \( x = 2 \) immediately gives \( C = \frac{1}{2} \), and setting \( x = 0 \) gives \( A = \frac{1}{4} \). Setting \( x = 1 \) gives

\[
1 = A - B + C,
\]

from which it is easy to get \( B = -\frac{1}{4} \). So

\[
\frac{1}{x^3 - 4x^2 + 4x} = \frac{1/4}{x} - \frac{1/4}{x-2} + \frac{1/2}{(x-2)^2}.
\]

(b) Integrating the answer from (a) gives

\[
\frac{1}{4} \ln |x| - \frac{1}{4} \ln |x-2| - \frac{1}{x-2}.
\]

5. (15 points)

Find the arc length of the parametric curve \( x(t) = e^t \cos t, \ y(t) = e^t \sin t \) connecting the point \((1,0)\) to the point \((-e^\pi,0)\).

Solution: First observe that the points \((1,0)\) and \((-e^\pi,0)\) correspond to \( t = 0 \) and \( t = \pi \), respectively. It follows that the arc length of this curve is given by

\[
L = \int_0^\pi \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \int_0^\pi \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} \, dt
\]

\[
= \int_0^\pi \sqrt{e^{2t} \cos^2 t - e^{2t} \sin t \cos t + e^{2t} \sin^2 t + e^{2t} \sin t \cos t + e^{2t} \cos^2 t} \, dt
\]

\[
= \int_0^\pi \sqrt{2e^{2t}(\cos^2 t + \sin^2 t)} \, dt
\]

\[
= \sqrt{2} \int_0^\pi e^t \, dt = \sqrt{2} [e^t]_0^\pi = \sqrt{2}(e^\pi - 1).
\]

6. (15 points)

Use the area formula in polar coordinates to find the area of the region that is both inside the circle \( x^2 + y^2 = 4 \) and to the right of the line \( x = 1 \).

Solution: In polar coordinates, the equation of the given circle is \( r = 2 \) and the equation of the given line is \( r \cos \theta = 1 \), or \( r = \sec \theta \). The circle and line intersect when \( \sec \theta = 2 \), or
\[ \cos \theta = \frac{1}{2}, \] which happens when \( \theta \) is \( \pi/3 \) or \( -\pi/3 \). By the area formula in polar coordinates, the area is

\[ \int_{-\pi/3}^{\pi/3} \frac{1}{2} \left( 2^2 - \sec^2 \theta \right) \, d\theta = \frac{1}{2} \left( 4\theta - \tan \theta \right) \bigg|_{-\pi/3}^{\pi/3} = \frac{1}{2} \left( 4\pi/3 - \tan(\pi/3) - (-4\pi/3 - \tan(-\pi/3)) \right). \]

So the area is \( 4\pi/3 - \sqrt{3} \).

**Part B**

7. (20 points)

(a) Find a power series representation centered at 0 of the function as well as the radius and interval of convergence.

\[ f(x) = \frac{x}{2 + x^2} \]

(b) Write the following function as a power series in \( x \). What is the radius of convergence of this power series?

\[ \frac{d}{dx} \left( \frac{x}{2 + x^2} \right) \]

**Solution:** (a)

\[
 f(x) = \frac{x}{2} \sum_{n=0}^{\infty} \left( -\frac{x^2}{2} \right)^n = \frac{x}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n+1}
\]

for \( \left| -\frac{x^2}{2} \right| < 1 \Leftrightarrow |x| < \sqrt{2} \).

So the radius of convergence \( R = \sqrt{2} \). Now we consider the boundary cases

\[ \left| -\frac{x^2}{2} \right| = 1 \Leftrightarrow x = \pm \sqrt{2}. \]

We can easily see that the series diverges by the divergence test. So, the interval of convergence is \( (-\sqrt{2}, \sqrt{2}) \).

(b)

\[
 \frac{d}{dx} \left( \frac{x}{2 + x^2} \right) = \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx} \frac{x^{2n+1}}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n 2n + 1}{2^{n+1}} \cdot x^{2n}
\]

for \( |x| < \sqrt{2} \) by the differentiation theorem. The radius of convergence is \( \sqrt{2} \) as well.

8. (20 points)
Find the radius of convergence and interval of convergence of the series
\[ \sum_{n=3}^{\infty} \frac{2^n(x+3)^n}{2n+1}. \]

**Solution:** We use the ratio test:

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{1}{a_n} \right| = \frac{2^{n+1}|x+3|^{n+1}}{2(n+1)+1} \cdot \frac{2n+1}{2^n|x+3|^n}
\]

as \( n \to \infty \). From

\[ 2|x+3| < 1 \Leftrightarrow |x+3| < \frac{1}{2}, \]

the radius of convergence \( R = \frac{1}{2} \).

Now consider the boundary case

\[ 2|x+3| = 1 \Leftrightarrow 2(x+3) = \pm 1 \Leftrightarrow x = -\frac{5}{2}, -\frac{7}{2}. \]

Plugging these in the original series expression, we get

\[ \sum_{n=3}^{\infty} \frac{(-1)^n}{2n+1}, \]

which diverges for +1 by limit comparison with \( \sum_{n=3}^{\infty} \frac{1}{2n} \) and converges for -1 by the Alternating series test. So the interval of convergence is \([-\frac{7}{2}, -\frac{5}{2})\).

9. (20 points)

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

\[ \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{(1+n^2) \cdot \tan^{-1} n} \]

**Solution:**

First, consider the series \( \sum_{n=1}^{\infty} \frac{n}{(1+n^2) \cdot \tan^{-1} n} \) for absolute convergence. Since

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

diverges by \( p \)-series test and

\[ \lim_{n \to \infty} \frac{n}{(1+n^2) \cdot \tan^{-1} n} \cdot n = \frac{2}{\pi}, \]

the series is divergent.
by the limit comparison test, the series diverges.

If we consider the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{(1+n^2) \tan^{-1} n} \), since \( \frac{n}{1+n^2} \) is positive and decreasing to 0 and \( \frac{1}{\tan^{-1} n} \) is positive and decreasing, \( \frac{n}{(1+n^2) \tan^{-1} n} \) is positive and decreasing to 0 and by the alternating series test, the series converges.

So, the series is conditionally convergent.

10. (20 points)

(a) Find the Taylor series centered at 0 of the function \( \cos \sqrt{|x|} \), as well as radius and interval of convergence.

(b) Write the integral \( \int_0^x \cos \sqrt{|t|} \, dt \) as a power series in \( x \).

Solution: (a) The Taylor series of \( \cos x \) is

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,
\]

which converges for all \( x \). Therefore, replacing \( x \) by \( \sqrt{|x|} \) gives,

\[
\cos \sqrt{|x|} = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{|x|})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots,
\]

which also converges for all \( x \).

(b)

\[
\int_0^x \cos \sqrt{|t|} \, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(2n)!} \, dt = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^n}{(2n)!} \, dt
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)(2n)!} \, dt
\]

\[
= x - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 4!} - \cdots = x - \frac{x^2}{4} + \frac{x^3}{72} - \cdots.
\]

The equation holds for all \( x \).

11. (20 points)
(a) Determine whether the series
\[ \sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \]
is absolutely convergent, conditionally convergent, or divergent.

(b) Estimate the sum of the series with an accuracy of \( \frac{1}{100} \).

**Solution:** a) Using the ratio test,

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left( \frac{(-1)^{n+1}}{(2n+3)(2n+3)!} \cdot \frac{2n+1}{(2n+1)(2n+1)!} \right) = \frac{-2n+1}{(2n+3)(2n+2)(2n+3)} = 0,
\]
so the series is absolutely convergent.

b) Since this is an alternating series, we wish to find \( n \) such that

\[ \left| \frac{(-1)^n}{(2n+1)(2n+1)!} \right| < \frac{1}{100} \]

This is false for \( n = 0, 1 \), but for \( n = 2 \) we have \( \frac{1}{5 \cdot 5!} = \frac{1}{600} < \frac{1}{100} \). Hence the sum

\[ s_1 = 1 - \frac{1}{3 \cdot 3!} = 1 - \frac{1}{18} = \frac{17}{18} \]

is accurate to within 1/100.