EXTRA CREDIT 6: DUE OCTOBER 23 AT 5PM

Question: Suppose you have two circles, one with radius r_1 centered at the origin, and one with radius r_2 centered at $(r_1 + r_2\,,\,0)$. Thus, the circles have the point $P = (r_1\,,\,0)$ in common. Suppose the circle with radius r_2 begins rolling counterclockwise along the outside of the other circle. As the circle rotates, the point that was originally at P will also rotate, tracing out a path in the plane. Find parametric equations for this path.

Solution: Suppose the outer circle has begun rotating and has reached a point in which the following two statements are true:

- ◇ The point the two circles have in common lies in the first quadrant.
- ◇ The point on the moving circle that was originally at P is now at some point $P_2 = (x, y)$ which lies to the right and below the center of the circle.

Let's call the center of the moving circle C, and the point the two circles have in common Q. Draw a line from the origin through Q and C. Let θ represent the angle this line makes with the positive x-axis. Also draw the line segment $\overline{CP_2}$ and let $\phi = \angle QCP_2$. The arcs PQ and QP_2 have the same length, which means:

$$r_1\theta = r_2\phi \qquad \Rightarrow \qquad \phi = \frac{r_1\theta}{r_2}$$

Now draw in a right triangle whose hypotenuse is $\overline{CP_2}$ and whose third vertex is a point above P_2 and to the right of C, call this P_3. Call the horizontal leg x_1 and the vertical leg y_1, and let $\alpha = \angle P_2CP_3$. Thus, we can now say:

$$\alpha = \pi - \phi - \theta = \pi - \frac{r_1\theta}{r_2} - \theta = \pi - (\frac{r_1 + r_2}{r_2})\theta$$

Using the addition and subtraction formula for sine and cosine, we obtain:

$$
\begin{aligned}
x_1 &= r_2 \cos\alpha \\
&= r_2 \cos(\pi - (\tfrac{r_1+r_2}{r_2})\theta) \\
&= r_2 \cos\pi \cos((\tfrac{r_1+r_2}{r_2})\theta) + r_2 \sin\pi \sin((\tfrac{r_1+r_2}{r_2})\theta) \\
&= -r_2 \cos((\tfrac{r_1+r_2}{r_2})\theta) \\
y_1 &= r_2 \sin\alpha \\
&= r_2 \sin(\pi - (\tfrac{r_1+r_2}{r_2})\theta) \\
&= r_2 \sin\pi \cos((\tfrac{r_1+r_2}{r_2})\theta) - r_2 \cos\pi \sin((\tfrac{r_1+r_2}{r_2})\theta) \\
&= r_2 \sin((\tfrac{r_1+r_2}{r_2})\theta)
\end{aligned}
$$

Write $C = (x_2, y_2)$, the center of the moving circle. Then we have $x_2 = (r_1 + r_2)\cos\theta$ and $y_2 = (r_1 + r_2)\sin\theta$. Recall that $P_2 = (x, y)$ is the point on the outer circle that we are keeping track of. From the information above, we now have our answer:

$$
\begin{aligned}
x &= x_1 + x_2 \\
&= (r_1 + r_2)\cos\theta - r_2 \cos((\tfrac{r_1+r_2}{r_2})\theta) \\
y &= y_2 - y_1 \\
&= (r_1 + r_2)\sin\theta - r_2 \sin((\tfrac{r_1+r_2}{r_2})\theta)
\end{aligned}
$$

Thus, parametric equations for the path are given by:

$$x = f(\theta) = (r_1 + r_2)\cos\theta - r_2 \cos((\frac{r_1+r_2}{r_2})\theta)$$

$$y = g(\theta) = (r_1 + r_2)\sin\theta - r_2 \sin((\frac{r_1+r_2}{r_2})\theta)$$

□