Math 162, Calculus IIA   Extra Credit 5   Name:

Please submit a solution to your instructor by Friday, October 16, 2007 before 5 pm.

1. ((1/N+1) points) Figuring out how many spheres can be packed into a container is an important problem with applications to such things as cell biology and the airline industry (not to mention the bowling industry!). A central problem is to find the density of the sphere packing, i.e., the ratio of the volume of the tightest sphere packing to the volume of the container. We propose the following simplified version of the sphere packing problem. Suppose that circles of equal diameter are packed tightly in n rows inside an equilateral triangle. (See, for example, page 762 [791 in the 5th edition] in your textbook.) If A is the area of the equilateral triangle and A_n is the total area occupied by the n rows of circles, show that
$$\lim_{n\to\infty} \frac{A_n}{A} = \frac{\pi}{2\sqrt{3}}.$$

Solution: Let s be the length of one of the sides of the equilateral triangle. Let r_n be the radius of the circle associated to the packing of the triangle by n rows of spheres. Along one of the sides of the triangle are n spheres. Let C_1 and C_2 be the centers of the two outermost spheres along one edge of the triangle. The distance between C_1 and C_2 is $2r_n(n-1)$. Dropping perpendiculars down from C_1 and C_2 to the side of the triangle gives two points D_1 and D_2. Let E_1 be the corner of the triangle nearest C_1 and D_1. Let x be the distance from E_1 to D_1. From the triangle $\Delta E_1C_1D_1$, we see that
$$\tan\left(\frac{\pi}{6}\right) = \frac{r_n}{x},$$
and thus $x = \sqrt{3}r_n$. The length of the side of the triangle is $s = 2x + 2r_n(n-1)$, and by substitution $s = 2\sqrt{3}r_n + 2r_n(n-1) = 2r_n(n + (\sqrt{3}-1))$. Since there are $1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$ spheres inside of the triangle,

$$\begin{aligned}
\lim_{n\to\infty} \frac{A_n}{A} &= \lim_{n\to\infty} \frac{\frac{n(n+1)}{2}\cdot \pi r_n^2}{s^2\frac{\sqrt{3}}{4}} \\
&= \lim_{n\to\infty} \frac{n(n+1)\pi r_n^2}{2}\cdot\frac{4}{s^2\sqrt{3}} \\
&= \lim_{n\to\infty} \frac{2n(n+1)\pi r_n^2}{(2r_n(n+(\sqrt{3}-1)))^2\sqrt{3}} \\
&= \lim_{n\to\infty} \frac{n(n+1)\pi}{2\sqrt{3}(n^2+2n(\sqrt{3}-1)+(\sqrt{3}-1)^2)} \\
&= \lim_{n\to\infty} \frac{\pi n^2+\pi n}{2\sqrt{3}n^2+4\sqrt{3}n(\sqrt{3}-1)+2\sqrt{3}(\sqrt{3}-1)^2} \\
&= \lim_{n\to\infty} \frac{\pi+\frac{\pi}{n}}{2\sqrt{3}+\frac{4\sqrt{3}(\sqrt{3}-1)}{n}+\frac{2\sqrt{3}(\sqrt{3}-1)^2}{n^2}} \\
&= \frac{\pi}{2\sqrt{3}} \approx 0.906899682117\ldots.
\end{aligned}$$