EXTRA CREDIT PROBLEM 4

DEADLINE: FRIDAY, OCTOBER 9 AT 5PM

Problem. Suppose $f(t)$ is a function defined on some interval $[a, b]$ whose derivative is continuous on the same interval. Use integration by parts to evaluate the following limit:

$$\lim_{\lambda \to \infty} \int_a^b f(t) \sin(\lambda t)\, dt$$

Solution. Fix $\lambda > 0$. Do integration by parts as follows:

$$u = f(t) \qquad dv = \sin(\lambda t)\, dt$$

$$du = f'(t)\, dt \qquad v = -\frac{1}{\lambda}\cos(\lambda t)$$

$$\begin{aligned}\int_a^b f(t)\sin(\lambda t)\, dt &= -\frac{1}{\lambda} f(t)\cos(\lambda t)]_a^b + \frac{1}{\lambda}\int_a^b f'(t)\cos(\lambda t)\, dt \\ &= \frac{1}{\lambda} f(a)\cos(\lambda a) - \frac{1}{\lambda} f(b)\cos(\lambda b) + \frac{1}{\lambda}\int_a^b f'(t)\cos(\lambda t)\, dt\end{aligned}$$

Now take absolute values and use the triangle inequality, as well as the fact that $|\cos x| \leq 1 \,\forall\, x \in \mathbb{R}$ to obtain the following:

$$\begin{aligned}\left|\int_a^b f(t)\sin(\lambda t)\, dt\right| &\leq \frac{1}{\lambda}\left|f(a)\cos(\lambda a)\right| + \frac{1}{\lambda}\left|f(b)\cos(\lambda b)\right| + \frac{1}{\lambda}\int_a^b \left|f'(t)\cos(\lambda t)\right|\, dt \\ &\leq \frac{1}{\lambda}\left|f(a)\right| + \frac{1}{\lambda}\left|f(b)\right| + \frac{1}{\lambda}\int_a^b \left|f'(t)\right|\, dt \\ &\leq \frac{3M}{\lambda}\end{aligned}$$

where $M = \max\{|f(a)|, |f(b)|, \int_a^b |f'(t)|\, dt\}$. Here we have used the fact that f' is continuous on $[a, b]$. Letting $\lambda \to \infty$ gives the desired result:

$$\lim_{\lambda \to \infty} \int_a^b f(t)\sin(\lambda t)\, dt = 0$$