EXTRA CREDIT PROBLEM 3
DEADLINE: 5PM FRIDAY, OCTOBER 2

Use integration by parts to show that for all $x > 0$,

$$0 < \int_0^\infty \frac{\sin t}{\ln(1+x+t)}\,dt < \frac{2}{\ln(1+x)}.$$

SOLUTION: We use integration by parts with

$$u = \frac{1}{\ln(1+x+t)} \qquad dv = \sin t\,dt$$
$$du = \frac{-1}{\ln^2(1+x+t)} \cdot \frac{1}{1+x+t}\,dt \qquad v = -\cos t$$

Then

$$\int_0^\infty \frac{\sin t\,dt}{\ln(1+x+t)} = \left.\frac{-\cos t}{\ln(1+x+t)}\right|_0^\infty - \int_0^\infty \frac{\cos t\,dt}{(1+x+t)\ln^2(1+x+t)} \tag{1}$$

First we note that for all real values of t, $-1 \le \cos t \le 1$ and $\cos t$ equals 1 or -1 only at integer multiples of π.

To evaluate the first term on the RHS of (1), we have

$$\left.\frac{-\cos t}{\ln(1+x+t)}\right|_0^\infty = \lim_{t\to\infty} \frac{-\cos t}{\ln(1+x+t)} + \frac{1}{\ln(1+x)}.$$

We use the Squeeze Theorem to compute the limit above.

$$\frac{-1}{\ln(1+x+t)} \le \frac{-\cos t}{\ln(1+x+t)} \le \frac{1}{\ln(1+x+t)}$$

and since

$$\lim_{t\to\infty} \frac{-1}{\ln(1+x+t)} = \lim_{t\to\infty} \frac{1}{\ln(1+x+t)} = 0,$$

our limit equals 0 and the first term in (1) is $\dfrac{1}{\ln(1+x)}$.

Now we find bounds for the second term on the RHS of (1). Since $-1 < \cos t < 1$ except at countably many points where $\cos t$ equals 1 or -1, we can bound the integral on the RHS of (1) in the following way:

$$-I < \int_0^\infty \frac{\cos t\,dt}{(1+x+t)\ln^2(1+x+t)} < I,$$

where

$$
\begin{aligned}
I &= \int_0^\infty \frac{1}{(1+x+t)\ln^2(1+x+t)}\,dt \\
&= \int -du \\
&= -u \\
&= -\frac{1}{\ln(1+x+t)}\Bigg|_0^\infty \\
&= \frac{1}{\ln(1+x)}.
\end{aligned}
$$

Therefore,

$$
\frac{1}{\ln(1+x)} - I < \int_0^\infty \frac{\sin t}{\ln(1+x+t)}\,dt < \frac{1}{\ln(1+x)} + I,
$$

and the result follows.