Extra credit problem 2

Find the volumes of a regular tetrahedraon, a regular octahedron and a regular cuboctahedron (look it up), each having edges of length s.

Show that a regular cuboctahedron is the disjoint (or nonoverlapping) union of eight regular octahedra and a number (to be determined by you) of isosceles right tetrahedra.

Solution

The volume $V(s)$ of any regular polyhedron with side s is cs^3, where c is a constant depending on the polyhedron. In the case of a cube, the constant is one. We need to find it for the tetrahedron, the octahedron and the cuboctahedron.

The regular tetrahedron

For the tetrahedron, consider the cube with side 2 centered at the origin with vertices at the eight points $(\pm 1, \pm 1, \pm 1)$. Its volume is 8. We can obtain a regular tetrahedron by removing the four right isosceles tetrahedra each with side 2 and vertices $(1,\ 1,\ 1)$, $(1,\ -1,\ -1)$, $(-1,\ 1,\ -1)$ and $(-1,\ -1,\ 1)$. (These are the four vertices having an even number of negative coordinates. We could also use the four vertices having an odd number of negative coordinates.) Each of them has volume $2^3/6 = 4/3$, so their total volume is $16/3$. Removing them from the cube leaves a regular tetrahedron with side $2\sqrt{2}$, the distance between and two of the remaining vertices. Thus we have

$$\begin{aligned} V\left(2\sqrt{2}\right) &= 8 - 16/3 = 8/3 \\ &= c\left(2\sqrt{2}\right)^3 = 16c\sqrt{2} \\ \text{so} \qquad c &= \frac{8}{3}\frac{1}{16\sqrt{2}} = \frac{1}{6\sqrt{2}} \\ &= \frac{\sqrt{2}}{12} \approx .1179. \end{aligned}$$

In particular the volume of a regular tetrahedron with side s is $s^3\sqrt{2}/12$.

The regular octahedron

We can construct a regular octahedron with side s as follows. Start with a regular tetrahedron with side $2s$ and "cut off its corners" by removing four regular tetrahedra each having sides s. The remaining solid is a regular octahedron with side s. The calculation above shwos that the volume of the larger tetrahedron is $8s^3\sqrt{2}/12 = 2s^3\sqrt{2}/3$. The volume of each small tetrahedron is 1/8 this amount, so the volume of the octahedron is 1/2 this amount, or $s^3\sqrt{2}/3$. Hence we have

$$c = \frac{\sqrt{2}}{3} \approx .4714$$

This is precisely four times the volume of a regular tetrahedron with side s. This is to be expected since we obtained the octahedron by removing four such tetrahera from a larger one with eight times the volume.

The regular cuboctahedron

A cuboctahedron can be obtained from a cube by cutting of its eight corners. More precisely, a regular cuboctahedron with side $s\sqrt{2}$ can be obtained by removing 8 isosceles right tetrahedra each with side s from a regular cube with side $2s$. The volume of the cube is $8s^3$ and that of each isosceles right tetrahedron is $s^3/6$. Hence we have

$$\begin{aligned} V(s\sqrt{2}) &= 8s^3 - \frac{8s^3}{6} = \frac{20s^3}{3} \\ \text{so} \qquad c &= \frac{20}{3}\frac{1}{\left(\sqrt{2}\right)^3} = \frac{5\sqrt{2}}{3} \approx 2.3570. \end{aligned}$$

This value is 20 times that for the regular tetrahedron.

Decomposition of the regular cuboctahedron

The cube above has vertices at the eight points $(\pm s, \pm s, \pm s)$, and the twelve vertices of the cuboctahedron are

$$(0, \pm s, \pm s), \quad (\pm s, 0, \pm s) \quad \text{and} \quad (\pm s, \pm s, 0).$$

It has six square faces centered at $(\pm s, 0, 0)$, $(0, \pm s, 0)$ and $(0, 0, \pm s)$.

Consider the bottom square face centered at $(0, 0, -s)$ and having vertices at the four points $(\pm s, 0, -s)$ and $(0, \pm s, -s)$. It is the base of a pyramid with apex at the origin $(0, 0, 0)$, which is the union of four isosceles right tetrahedra with side s. The pyramid's volume is therefore $2s^3/3$.

There are similar pyramids based at each of the other square faces each having its apex at the origin. The total volume of the six pyramids, which consist of 24 isosceles right tetrahedra with side s, is $4s^3$. Removing them from the cuboctahedron leaves a solid with volume of $8s^3/3$, which is the union of eight regular tetrahedra with side $s\sqrt{2}$.