Solution to extra credit problem 1

Find the area of the region inside a regular hexagon with side 2 consisting of points that are closer to the center of the hexagon than to any of its sides.

The set of points equidistant from a given line (eg one side of the hexagon) and a given point (eg the center of the hexagon) is a certain parabola, so the boundary of our region is the union of six parabolic arcs.

We can take the vertices of the hexagon to be $(\pm 2, 0)$ and $(\pm 1, \pm\sqrt{3})$. Our region has 12 symmetric subregions, one of which is bounded by the lines $x = 0$, $y = x\sqrt{3}$, and the parabola determined by the origin and the top edge of the hexagon, which is the line defined by $y = \sqrt{3}$. The equation for this parabola is

$$\begin{aligned} \sqrt{x^2+y^2} &= \sqrt{3} - y \\ x^2 + y^2 &= \left(\sqrt{3} - y\right)^2 \\ &= 3 - 2\sqrt{3}y + y^2 \\ y &= \frac{3 - x^2}{2\sqrt{3}}. \end{aligned}$$

This parabola meets the line $y = x\sqrt{3}$ at the point $\left(2\sqrt{3} - 3, 6 - 3\sqrt{3}\right)$. The area of this region is 1/12 the area we are seeking, so the latter is

$$\begin{aligned} A &= 12 \int_0^{2\sqrt{3}-3} \left(\frac{3 - x^2}{2\sqrt{3}} - x\sqrt{3}\right) \mathrm{d}x \\ &= \frac{12}{2\sqrt{3}} \int_0^{2\sqrt{3}-3} \left(3 - x^2 - 6x\right) \mathrm{d}x \\ &= 2\sqrt{3}\left(3x - \frac{x^3}{3} - 3x^2\right)\Bigg|_0^{2\sqrt{3}-3} \\ &= 2\sqrt{3}\left(-27 + 16\sqrt{3}\right) \\ &= 96 - 54\sqrt{3} \\ &\approx 2.46926. \end{aligned}$$

The area of the hexagon is $6\sqrt{3} \approx 10.39231$, slightly more than four times the area of the inner region.

The boundary of the region consists of six parabolic arcs whose equations are

$$y = \pm\frac{3 - x^2}{2\sqrt{3}} \qquad \text{and} \qquad \left(2\sqrt{3} \pm x\sqrt{3} \pm y\right)^2 = 4x^2 + 4y^2.$$