Math 161 **Spring 2003**

Review - 3

THEORY

L'Hospital's Rule

Indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$:
if

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad \text{or} \qquad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note: you may need to use this rule twice or even more times.
Convert indeterminate forms of types $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞ into quotients:

$$f(x)\cdot g(x) = \frac{f(x)}{g(x)^{-1}} \qquad \text{or} \qquad \frac{g(x)}{f(x)^{-1}}$$

Note: sometimes only one of these works. There may be no way to tell which one will work. Try one, if it doesn't bring you to any answer, try the other one.

$$f(x)^{g(x)} = e^{(\ln f(x))g(x)}, \qquad \lim_{x\to a} e^{(\ln f(x))g(x)} = e^{\left(\lim_{x\to a} (\ln f(x))g(x)\right)} = e^{\left(\lim_{x\to a} \frac{\ln f(x)}{g(x)^{-1}}\right)}$$

Example 1. Evaluate $\lim_{x\to 0} \frac{\sin(3x)}{e^x - 1}$

Solution: $\lim_{x\to 0} \sin(3x) = 0$ and $\lim_{x\to 0} e^x - 1 = 0$, so we use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\sin(3x)}{e^x - 1} = \lim_{x\to 0} \frac{(\sin(3x))'}{(e^x-1)'} = \lim_{x\to 0} \frac{3\cos(3x)}{e^x} = \frac{3\cdot 1}{1} = 3$$

Example 2. Evaluate $\lim_{x\to 0^+} \sin x^{\tan x}$

Solution: $\lim_{x\to 0^+} \sin x^{\tan x} \ (= 0^0) = \lim_{x\to 0^+} \left(e^{\ln(\sin x)}\right)^{\tan x} = \lim_{x\to 0^+} e^{(\ln(\sin x))\tan x} =$

$$= e^{\left(\lim_{x\to 0^+} (\ln(\sin x))\tan x\right)}$$

$$\lim_{x\to 0^+} (\ln(\sin x))\tan x \ \ (= -\infty\cdot 0) = \lim_{x\to 0^+} \frac{\ln(\sin x)}{(\tan x)^{-1}} \ \left(= \frac{-\infty}{\infty}\right) = \lim_{x\to 0^+} \frac{(\ln(\sin x))'}{((\tan x)^{-1})'} =$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cos x}{-(\tan x)^{-2}\sec^2 x} = \lim_{x\to 0^+} \frac{(\tan x)^2\cos^3 x}{-\sin x} = \lim_{x\to 0^+} \frac{\sin^2 x\cos^3 x}{-\sin x\cos^2 x} =$$

$$= \lim_{x\to 0^+} (-\sin x\cos x) = 0.$$

Therefore, $\lim_{x\to 0^+} \sin x^{\tan x} = e^0 = 1$

Summary of curve sketching

Def. The domain of a function $f(x)$ is the set of all values of x for which the function is defined.

To find the x-intercepts, solve $f(x) = 0$ for x. The y-intercept is $f(0)$.

Def. $f(x)$ is called even if $f(x) = f(-x)$ for all x. $f(x)$ is called odd if $f(x) = -f(-x)$ for all x. E.g. $\cos x$ is even, and $\sin x$ is odd. $f(x)$ is called periodic if there exists a number p such that $f(x) = f(x+p)$ for all x. The smallest positive number p such that $f(x) = f(x+p)$ for all x is called the period of $f(x)$. E.g. $\cos x$ and $\sin x$ are periodic with period 2π.

Def. If either $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ (or both), then $y = L$ is called a horizontal asymtote. If either $\lim_{x\to a^+} = \pm\infty$ or $\lim_{x\to a^-} = \pm\infty$ (or both), then $x = a$ is a vertical asymptote.

Def. A number c in the domain of $f(x)$ is called a critical number of $f(x)$ if either $f'(c)$ doesn't exist or $f'(c) = 0$.

Increading/Decreasing test. $f(x)$ is increasing when $f'(x) > 0$, and decreasing when $f'(x) < 0$.

First derivative test. If $f'(x)$ changes from positive to negative at c, then c is a local maximum for $f(x)$. If $f'(x)$ changes from negative to positive at c, then c is a local minimum for $f(x)$. (In both cases c must be in the domain of $f(x)$, and $f(x)$ must be continuous at c.)

Concavity test. $f(x)$ is CU (concave upward) when $f''(x) > 0$, and CD (concave down) when $f''(x) < 0$.

Def. If $f(x)$ changes the direction of concavity at c (and c is in the domain of $f(x)$), then c is an inflection point.

Curve sketching. Draw asymptotes as dashed lines. Plot all the intercepts, local maximum and minimum points, and inflection points. Then draw the graph of the function, making it pass throught all the points, approach the asymptotes, fall and rise according to the increasing/decreasing test, and with concavity according to the concavity test. If the function is even, odd, or periodic, use this fact (the graph of an even function is symmetric about the y-axis, the graph of an odd function is symmetric about the origin, and the graph of a periodic function is periodic).

Example. There is a separate handout with an example, $f(x) = \dfrac{2x^2}{x^2-1}$. Take a copy from the box outside of my office if you were not in class when this example was done.

Optimization problems

1. Draw a picture whenever possible.
2. Introduce notations. Assign symbols to the quantities that you need to find, and to the quantity that you want to maximize or minimize.
3. Express the quantity that is to be maximized or minimized in terms of some other quantities.
4. Use the given information to find relationships among the unknown quantities.
5. Express the quantity that is to be maximized or minimized in terms of just one variable.
6. To find a local maximum or minimum, differentiate the function from step 5, set the derivative equal to 0, and solve for the variable.
7. Find the values of all the required quantities.

Example. Find the point on the line $2x + y - 5 = 0$ and in the first quadrant such that the area of the rectangle bounded by the horizontal and the vertical lines through this point, the x-axis, and the y-axis, is as lagre as possible.

The area of the rectangle is $A = xy$, and since the point must lie on the above line, we have $y = 5 - 2x$. Then the area can be expressed as a function of one variable x, namely $A(x) = x(5 - 2x) = 5x - 2x^2$. Differentiate and set equal to 0: $A'(x) = 5 - 4x = 0$, so $x = 1.25$. Obviously, the derivative changes from positive to negative at 1.25, so we have a maximum. The y-coordinate of this point is $5 - 2 \cdot 1.25 = 2.5$.

There are 5 very good examples in the book (section 4.7). I strongly recommend you to read them all.

Newton's method

To approximate a root of an equation $f(x) = 0$, choose an appropriate initial approximation (e.g. sketch the graph of $f(x)$ and choose a number which is close to the root), call this initial approximation x_1. Then find x_2, x_3, etc. using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Antiderivatives and indefinite integrals

Def. $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$.

If $F(x)$ is an antiderivative of $f(x)$, then for any constant c, $F(x) + c$ is also an antiderivative of $f(x)$. Also, any antiderivative of $f(x)$ has the form $F(x) + c$. So the family of functions $F(x) + c$ is called the most general antiderivative of $f(x)$, and it is also called the indefinite integral of $f(x)$ denoted by $\int f(x)dx$.

Here is a table of indefinite integrals:

$f(x)$	$\int f(x)dx$
a	$ax + c$
$x^n, \quad n \neq -1$	$\dfrac{x^{n+1}}{n+1} + c$
x^{-1}	$\ln\lvert x\rvert + c$
e^x	$e^x + c$
a^x	$\dfrac{a^x}{\ln a} + c$

$f(x)$	$\int f(x)dx$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\csc^2 x$	$-\cot x + c$
$\sec x \tan x$	$\sec x + c$
$\csc x \cot x$	$-\csc x + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$
$\dfrac{1}{1+x^2}$	$\arctan x + c$

The following rules correspond to the sum, difference, and constant multiple rules for derivatives:

$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$
$\int (cf(x))dx = c \int f(x)dx$

If you are asked to give the particular anitiderivative that satisfies a certain initial contion, find the most general antiderivative (with c) first, and then use the initial condition to find the constant c. If you are given the second derivative $f''(x)$ (and possibly two initial condtions), then you'll have to antidifferentiate twice in order to find the function $f(x)$.

For example, to find $f(x)$ such that $f''(x) = \cos x$, $f'(0) = 1$ and $f(0) = 2$, write $f'(x) = \sin x + c$. The condition $f'(0) = 1$ implies that $c = 1$, then $f'(x) = \sin x + 1$, and $f(x) = -\cos x + x + d$. Now the condtion $f(0) = 2$ implies that $d = 3$, therefore $f(x) = -\cos x + x + 3$.

For an object moving along a straight line, its velocity function is an antiderivative of its acceleration function, and its position function is an antiderivative of its velocity function.

Note: acceleration due to gravity is approximately 9.8 $\mathrm{m/s^2}$, or 32 $\mathrm{ft/s^2}$.

Area and the definite integral

Def. Let $f(x)$ be a continuous function on $[a,b]$. Divide $[a,b]$ into n subintervals of equal lenght $\Delta x = \dfrac{b-a}{n}$. Let $x_0 = a,\ x_1,\ x_2,\ \dots,\ x_{n-1},\ x_n = b$ be the endpoints of these subintervals. Choose a sample point in each subinterval: $x_i^* \in [x_{i-1}, x_i]$. Then the sum $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ is called a Riemann sum, and a limit of it as n approaches infinity is called the integral of $f(x)$ from a to b:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Note: If $f(x) > 0$, then the Riemann sum represents the sum of areas of approximating rectangles, and the integral $\int_a^b f(x)dx$ represents the area of the region under the graph of $f(x)$. If $f(x)$ takes on both positive and negative values, then $\int_a^b f(x)dx$ is the sum of the areas of regions under the graph of $f(x)$ and avove the x-axis minus the sum of the areas of regions above the graph of $f(x)$ and below the x-axis.

Example. The value of the integral $\int_{-4}^{6} \frac{x}{2}dx$ is the area of triangle 1 minus the area of triangle 2, i.e. $\frac{1}{2}\cdot 6\cdot 3 - \frac{1}{2}\cdot 4\cdot 2 = 9 - 4 = 5.$

Fundamental theorem of calculus

Part I. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

Part II. If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Example. $\displaystyle\int_{-4}^{6} \frac{x}{2}dx = \frac{x^2}{4}\Big|_{-4}^{6} = \frac{6^2}{4} - \frac{(-4)^2}{4} = \frac{36}{4} - \frac{16}{4} = 9 - 4 = 5.$

Cor. If $g(x) = \displaystyle\int_{a(x)}^{b(x)} f(t)dt$, then $g'(x) = f(b(x))b'(x) - f(a(x))a'(x)$

Example. $\displaystyle\frac{d}{dx}\int_{\sin x}^{\cos x} \sqrt{t}dt = \sqrt{\cos x}(-\sin x) - \sqrt{\sin x}\cos x = -\sqrt{\cos x}\sin x - \sqrt{\sin x}\cos x.$

The substitution rule

For an integral of the form $\int f(g(x))g'(x)dx$ make the substitution $u = g(x)$, then $du = g'(x)dx$, and

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Evaluate the integral $\int f(u)du$ (let $F(u)$ be an antiderivative of $f(u)$), and change back to the original variable x:

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + c = F(g(u)) + c$$

For a definite integral there are 2 ways to use substitution:

1. Evaluate the corresponding indefinite integral first, change back to the original variable, and then use the old limits of integration:

   If $\int f(g(x))g'(x)dx = \int f(u)du = F(u) + c = F(g(u)) + c$,

   then $\int_a^b f(g(x))g'(x)dx = F(g(u))\Big|_a^b = F(g(b)) - F(g(a))$.

2. Change the limits of integration as follows: since $u = g(x)$, $u = g(a)$ when $x = a$, and $u = g(b)$ when $x = b$, thus

   $$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).$$

Example. $\int_0^{\frac{\pi}{2}} \sin(2x)dx$

let $u = 2x$, then $du = 2dx$, or $\frac{du}{2} = dx$,

and the new limits of integration are $2 \cdot 0 = 0$ and $2 \cdot \frac{\pi}{2} = \pi$,

so the integral becomes

$$\int_0^{\pi} \frac{\sin u}{2} du = \frac{1}{2}\int_0^{\pi} \sin u du = \frac{1}{2}(-\cos u)\Big|_0^{\pi} = \frac{1}{2}(-\cos\pi - (-\cos 0)) = \frac{1}{2}(1 - (-1)) = 1.$$

There are nine very good examples in the book, section 5.5. I recommend that you look at them.