Math 161

Review - 2

THEORY

Derivatives

Definition. The derivative of $f(x)$ at a point a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

(if this limit exists. In this case we say that $f(x)$ is differentiable at a. If the limit does not exist, then $f(x)$ is not differentiable at a.)

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Differentiation rules

1. Sum and difference rules $(f+g)' = f' + g'$, $(f-g)' = f' - g'$
2. Constant multiple rule $(cf)' = cf'$ for any constant c
3. Product rule $(fg)' = f'g + fg'$
4. Quotient rule $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
5. Chain rule $(f \circ g)'(x) = (f(g(x)))' = f'(g(x))g'(x)$

Derivatives of elementary functions

$$(x^n)' = nx^{n-1}, \qquad (e^x)' = e^x, \qquad (a^x)' = (\ln a)a^x,$$

$$(c)' = 0, \qquad (\ln x)' = \frac{1}{x}, \qquad (\log_a x)' = \frac{1}{(\ln a)x},$$

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad (\tan x)' = (\sec x)^2,$$

$$(\csc x)' = -\csc x \cot x, \qquad (\sec x)' = \sec x \tan x, \qquad (\cot x)' = -(\csc x)^2,$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \qquad (\arctan x)' = \frac{1}{x^2+1}$$

Implicit differentiation

is differentiation without solving for y. Be sure to use the chain rule when you differentiate a function of $y = y(x)$ with respect to x:

$$(f(y))' = (f(y(x)))' = f'(y(x)) \cdot y'(x)$$

Example.
If $\sqrt[3]{xy} = x^2y - 7x$ and $y(2) = 4$, find $y'(2)$.
Solution:
First rewrite using $y(x)$ instead of just y: $\sqrt[3]{xy(x)} = x^2y(x) - 7x$

$$\frac{1}{3}(xy(x))^{-2/3}(1 \cdot y(x) + xy'(x)) = 2xy(x) + x^2y'(x) - 7$$

$$\frac{y(x)}{3(xy(x))^{2/3}} + \frac{xy'(x)}{3(xy(x))^{2/3}} = 2xy(x) + x^2y'(x) - 7$$

$$\frac{xy'(x)}{3(xy(x))^{2/3}} - x^2y'(x) = 2xy(x) - 7 - \frac{y(x)}{3(xy(x))^{2/3}}$$

$$y'(x)\left(\frac{x}{3(xy(x))^{2/3}} - x^2\right) = 2xy(x) - 7 - \frac{y(x)}{3(xy(x))^{2/3}}$$

$$y'(x) = \frac{2xy(x) - 7 - \frac{y(x)}{3(xy(x))^{2/3}}}{\left(\frac{x}{3(xy(x))^{2/3}} - x^2\right)}$$

if $x = 2$ and $y(2) = 4$, $\quad y'(2) = \dfrac{2 \cdot 2 \cdot 4 - 7 - \frac{4}{3(2\cdot4)^{2/3}}}{\left(\frac{2}{3(2\cdot4)^{2/3}} - 2^2\right)} = -\dfrac{52}{23}$

Higher derivatives

Definition. Let $y = f(x)$. The derivative of the derivative of $f(x)$ is called the second derivative of $f(x)$:

$$y'' = f''(x) = (y')' = (f'(x))' =$$
$$= \frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d^2}{(dx)^2}(f(x)) = \frac{d^2f(x)}{(dx)^2} = \frac{d^2y}{dx^2}$$

The derivative of the second derivative is called the third derivative:

$$f'''(x) = (f''(x))'$$

In general, the n-th derivative of $f(x)$ is the derivative of the $(n-1)$-st derivative of $f(x)$:

$$f^{(n)}(x) = (f^{(n-1)}(x))'$$

Logarithmic differentiation

is helpful for differentiating complicated functions involving products, quotients, or powers.

1. Take natural logarithm of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify the right-hand side.

$$\ln(AB) = \ln A + \ln B \qquad \ln\left(\frac{A}{B}\right) = \ln A - \ln B \qquad \ln(A^r) = r\ln A$$

2. Differentiate implicitly with respect to x
(Remember to use the chain rule on the left: $(\ln y)' = \frac{1}{y}\cdot y'$).

3. Sove the resulting equation for y'.

Example 1.
Differentiate $y = \dfrac{(\sin^2 x)\sqrt{\tan x}}{(x^2+1)^{3x}}$
Solution:

$$\ln y = \ln\frac{(\sin^2 x)\sqrt{\tan x}}{(x^2+1)^{3x}} = \ln((\sin^2 x)\sqrt{\tan x}) - \ln(x^2+1)^{3x} =$$

$$= \ln(\sin^2 x) + \ln\sqrt{\tan x} - \ln(x^2+1)^{3x} = 2\ln(\sin x) + \frac{1}{2}\ln(\tan x) - 3x\ln(x^2+1)$$

$$\frac{1}{y}\cdot y' = \frac{2}{\sin x}\cos x + \frac{1}{2\tan x}\sec^2 x - \left(3\ln(x^2+1) + 3x\frac{1}{x^2+1}2x\right)$$

$$y' = y\left(\frac{2\cos x}{\sin x} + \frac{\sec^2 x}{2\tan x} - 3\ln(x^2+1) - \frac{6x^2}{x^2+1}\right)$$

$$y' = \frac{(\sin^2 x)\sqrt{\tan x}}{(x^2+1)^{3x}}\left(\frac{2\cos x}{\sin x} + \frac{\sec^2 x}{2\tan x} - 3\ln(x^2+1) - \frac{6x^2}{x^2+1}\right)$$

Example 2.
Differentiate $y = 3(\ln x)^{\cos x}$
Solution:

$$\ln y = \ln\left(3(\ln x)^{\cos x}\right) = \ln 3 + \ln\left((\ln x)^{\cos x}\right)$$

$$\ln y = \ln 3 + \cos x\cdot\ln(\ln x)$$

$$\frac{1}{y}\cdot y' = (-\sin x)\ln(\ln x) + \cos x\frac{1}{\ln x}\cdot\frac{1}{x}$$

$$y' = y\left(-\sin x\cdot\ln(\ln x) + \frac{\cos x}{(\ln x)x}\right)$$

$$y' = 3(\ln x)^{\cos x}\left(-\sin x\cdot\ln(\ln x) + \frac{\cos x}{x\ln x}\right)$$

Related rates

Strategy:

1. Read the problem carefully. Draw a diagram if possible.
2. Introduce notation. Assign symbols to all quantities that are functions of time.
3. Express the given information and the requited rate in terms of derivatives.
4. Write an equation that relates the various quantities of the problem.
5. Differentiate both sides of the equation with respect to t.
   (Usually you will need the chain rule. Say if x is a function of t, then $\frac{d}{dt}(f(x)) = f'(x) \cdot \frac{dx}{dt}$.)
6. Substitute the given information into the resulting equation and solve for the unknown rate.

Example.

A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 3 km from P?

Solution:

1.

2. Let θ denote the angle between the beam of light LB and LP, and $x = |BP|$.
3. Since the light makes four revolutions ($=4 \cdot 2\pi$) per minute, the rate of change of θ is $\frac{d\theta}{dt} = 8\pi/\text{min}$.
We need to find $\frac{dx}{dt}$ at when $x = 3$.
4. $\tan\theta = \frac{x}{3} \quad \Rightarrow \quad x = 3\tan\theta$
5. Differentiate both sides with respect to t:
$\frac{dx}{dt} = 3\sec^2\theta \cdot \frac{d\theta}{dt}$
6. When $x = 3$, $\theta = \frac{\pi}{4}$, and
$\frac{dx}{dt} = 3\sec^2\frac{\pi}{4} \cdot 8\pi = 3 \cdot 2 \cdot 8\pi = 48\pi$ (km/min)

Linear approximations and differentials

Let $y = f(x)$. The curve lies very close to its tangent line near the point of tangency. An equation of the tangent line at the point $(a, f(a))$ is $y - f(a) = f'(a)(x-a)$,
or $y = f(a) + f'(a)(x-a)$,
so $f(x) \approx f(a) + f'(a)(x-a)$
is called the linear approximation of f at a.

Def. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x-a)$$

is called the linearization of f at a.

Def. $dx = \Delta x$ is the change in x.

$\Delta y = f(a + \Delta x) - f(a)$ represents the amount that the curve $y = f(x)$ rises or falls (the change in $f(x)$) when x changes by an amount dx.

$dy = f'(a)dx$ represents the amount that the tangent line rises or falls (the change in the linearization) when x changes by an amount dx.

Example.

(a) Find the linearization $L(x)$ of $y = x^2 + 2x$ at $a = 1$ and use it to approximate the value of the function at 1.05.

(b) Compute Δy and dy for $x = 1$ and $dx = 0.05$.

Solution:

(a) $f'(x) = (x^2 + 2x)' = 2x + 2$
$f(1) = 3, \quad f'(1) = 4$
$L(x) = f(1) + f'(1)(x-1) = 3 + 4(x-1) = 4x - 1$
Linearization: $L(x) = 4x - 1$.
$L(1.05) = 4 \cdot 1.05 - 1 = 3.2$
Therefore, $f(1.05) \approx 3.2$

(b) $\Delta y = f(1.05) - f(1) = 3.2025 - 3 = 0.2025$
$dy = f'(a)dx = f'(1) \cdot 0.05 = 4 \cdot 0.05 = 0.2$