Math 161 **Spring 2003**

Review - 3

SOLUTIONS

1. Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{\sin x}{x^3} \quad \left(= \frac{0}{0}\right) \quad = \lim_{x\to 0} \frac{(\sin x)'}{(x^3)'} = \lim_{x\to 0} \frac{\cos x}{3x^2} \quad \left(= \frac{1}{\text{small pos.}}\right) \quad = +\infty$

(b) $\lim_{x\to\infty} \frac{\ln(\ln x)}{x} \quad \left(= \frac{\infty}{\infty}\right) \quad = \lim_{x\to\infty} \frac{\frac{1}{\ln x}\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\ln x} \quad \left(= \frac{0}{0}\right) \quad = \lim_{x\to\infty} \frac{-\frac{1}{x^2}}{\frac{1}{x}} =$
$= \lim_{x\to\infty} \left(-\frac{1}{x}\right) = 0$

(c) $\lim_{x\to 1} (x-1)\tan\left(\frac{\pi x}{2}\right) \quad (= 0\cdot\infty) \quad = \lim_{x\to 1} \frac{(x-1)\sin\left(\frac{\pi x}{2}\right)}{\cos\left(\frac{\pi x}{2}\right)} \quad \left(= \frac{0}{0}\right)$
$= \lim_{x\to 1} \frac{\sin\left(\frac{\pi x}{2}\right) + (x-1)\cos\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}}{-\sin\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}} = \frac{1+0}{-\frac{\pi}{2}} = -\frac{2}{\pi}$

(d) $\lim_{x\to 0} \left(\frac{1}{x} - \csc x\right) \quad (= \infty - \infty) \quad = \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x}{x\sin x} \quad \left(= \frac{0}{0}\right)$
$= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} \quad \left(= \frac{0}{0}\right) \quad = \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$

(e) $\lim_{x\to 0^+} (-\ln x)^x \quad \left(= \infty^0\right) \quad = \lim_{x\to 0^+} \left(e^{\ln(-\ln x)}\right)^x = \lim_{x\to 0^+} e^{x\ln(-\ln x)} =$
$e^{\lim_{x\to 0^+} x\ln(-\ln x)} = e^{\lim_{x\to 0^+} \frac{\ln(-\ln x)}{\frac{1}{x}}} \quad \left(= e^{\frac{\infty}{\infty}}\right) \quad = e^{\lim_{x\to 0^+} \frac{\frac{1}{-\ln x}\cdot\left(-\frac{1}{x}\right)}{-\frac{1}{x^2}}}$
$= e^{\lim_{x\to 0^+} \left(-\frac{x}{\ln x}\right)} = e^0 = 1$

2. Let $f(x) = \frac{x}{(1+x)^2}$. Find the following:

(a) domain
$f(x)$ is defined for all x except -1, therefore, the domain is $(-\infty, -1) \cup (-1, +\infty)$.

(b) intercepts
To find x-intercepts, solve $f(x) = 0$ for x: $\quad \frac{x}{(1+x)^2} = 0 \quad x = 0.$
The y-intercept is $f(0) = \frac{0}{(1+0)^2} = 0$, so the only intercept is $(0,0)$.

(c) asymptotes
Horizontal asymptotes:
$\lim_{x\to+\infty} \frac{x}{(1+x)^2} = \lim_{x\to+\infty} \frac{x}{x^2+2x+1} = \lim_{x\to+\infty} \frac{1}{2x+2} = 0$

$$\lim_{x\to-\infty}\frac{x}{(1+x)^2}=\lim_{x\to-\infty}\frac{x}{x^2+2x+1}=\lim_{x\to-\infty}\frac{1}{2x+2}=0$$

Thus there is one horizontal asymptote $y=0$.

Vertical asymptotes:

$$\lim_{x\to-1^+}\frac{x}{(1+x)^2}\left(=\frac{-1}{\text{small positive}}\right)=-\infty$$

$$\lim_{x\to-1^-}\frac{x}{(1+x)^2}\left(=\frac{-1}{\text{small positive}}\right)=-\infty$$

Thus $x=-1$ is a vertical asymptote.

(d) critical numbers

$$f'(x)=\frac{1(1+x)^2-x2(x+1)}{(1+x)^4}=\frac{(1+x)-2x}{(1+x)^3}=\frac{1-x}{(1+x)^3}.$$

$f'(x)$ is not defined only at $x=-1$, but -1 is not in the domain of $f(x)$; $f'(x)=0$ at $x=1$, so 1 is the only critical number.

(e) intervals of increase and decrease

$f(x)$ is increasing when $f'(x)>0$, and decreasing when $f'(x)<0$.

	$-$	$+$	$-$
$f'(x)$	——— -1	——— 1	———

Therefore $f(x)$ is increasing on $(-1,1)$ and decreasing on $(-\infty,-1)$ and $(1,+\infty)$.

(f) local and absolute maxima and minima

1 is a local maximum because the derivative changes from positive to negative at 1.

Even though the derivative changes from neg. to pos. at -1, it is not a local minimum because $f(-1)$ is undefined.

There is no absolute minimum because $\lim_{x\to-1^+}\frac{x}{(1+x)^2}=\lim_{x\to-1^-}\frac{x}{(1+x)^2}=-\infty$.

1 is an absolute maximum because it is the only critical number, and there are no vertical asymptotes with $\lim_{x\to a}\frac{x}{(1+x)^2}=\infty$. The absolute maximum value is $f(1)=\frac{1}{4}$.

(g) intervals of concavity

$$f''(x)=\frac{(-1)(1+x)^3-(1-x)3(1+x)^2}{(1+x)^6}=\frac{-(1+x)-3(1-x)}{(1+x)^4}=\frac{2x-4}{(1+x)^4}.$$

$f''(x)>0$ when $x>2$, and $f''(x)<0$ when $x<2$, therefore $f(x)$ is CU on $(2,+\infty)$, and CD on $(-\infty,-1)\cup(-1,2)$.

(h) inflection points

$x=2$ is the only inflection point ($f(x)$ changes from CD to CU at 2).

(i) sketch the graph of $f(x)$

3. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side 10 if one side of the rectangle lies on the base of the triangle.

We want to find x and y such that the area of the rectangle, i.e. $2xy$, is a maximum.

Use similar triangles to find a relationship between x and y, e.g.

$\frac{5-x}{5} = \frac{y}{h}$ where $h = 5\tan 60^0 = 5\sqrt{3}$, so we have

$\frac{5-x}{5} = \frac{y}{5\sqrt{3}}$

Multiplying both sides by $5\sqrt{3}$ gives $\sqrt{3}(5-x) = y$.

Now we can express the area as a function of one variable x:

$A(x) = 2x\sqrt{3}(5-x) = 10\sqrt{3}x - 2\sqrt{3}x^2$. To find a maximum, we have to differentiate $A(x)$ and set the derivative equal to 0:

$A'(x) = 10\sqrt{3} - 4\sqrt{3}x = 0$

$10\sqrt{3} = 4\sqrt{3}x$

$x = 2.5$

It is easy to see that $A'(x)$ changes from positive to negative at 2.5, so this is a local maximum.

$y = \sqrt{3}(5-x) = 2.5\sqrt{3}$, thus the width of the rectangle is $2x = 2 \cdot 2.5 = 5$, and the height is $y = 2.5\sqrt{3}$.

4. Use Newton's method to approximate the root of the equation $x^2 - 23 = 0$. Choose a reasonable initial approximation x_1, and use it to find the second approximation x_2.

   A reasonable approximation is $x_1 = 5$ because $5^2 = 25$ which is close to 23.

   Let $f(x) = x^2 - 23$, then $f'(x) = 2x$, and by Newton's method

   $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 5 - \frac{2}{10} = 4.8.$

5. Find $f(x)$ if

   (a) $f'(x) = 1 - 8x^3 + 2\sin x - \cos x + 3e^x$, $f(0) = 5$.
   $f(x) = x - 2x^4 - 2\cos x - \sin x + 3e^x + c$
   $f(0) = -2 + 3 + c = 5 \quad \Rightarrow \quad c = 4$
   Therefore $f(x) = x - 2x^4 - 2\cos x - \sin x + 3e^x + 4$

   (b) $f''(x) = 6 - 24x^2$, $f'(1) = -3$, $f(2) = -32$.
   $f'(x) = 6x - 8x^3 + c$
   $f'(1) = 6 - 8 + c = -3 \quad \Rightarrow \quad c = -1$
   $f'(x) = 6x - 8x^3 - 1$
   $f(x) = 3x^2 - 2x^4 - x + d$
   $f(2) = 12 - 32 - 2 + d = -32 \quad \Rightarrow \quad d = -10$
   $f(x) = 3x^2 - 2x^4 - x - 10$

6. Evaluate the following definite integrals (using the fundamental theorem of calculus or by interpreting the integral in terms of areas)

   (a) $\displaystyle\int_1^3 (3x^2 - 6x + 5)dx = (x^3 - 3x^2 + 5x)\Big|_1^3 = (27 - 27 + 15) - (1 - 3 + 5) =$
   $15 - 3 = 12$

   (b) $\displaystyle\int_2^8 |x - 4|dx = A_1 + A_2 = \frac{1}{2} \cdot 2 \cdot 2 + \frac{1}{2} \cdot 4 \cdot 4 = 2 + 8 = 10$

   (c) $\displaystyle\int_\pi^{3\pi} \cos x dx = \sin x\Big|_\pi^{3\pi} = \sin(3\pi) - \sin(\pi) = 0 - 0 = 0$
   Note: also, it is obvious from the graph that $A_+ - A_- = 0$.

(d) $\int_{-1}^{2} \sqrt{4-s^2}ds = A_1 + A_2 = \frac{1}{2} \cdot 1 \cdot \sqrt{3} + \frac{1}{3} \cdot \pi \cdot 2^2 = \frac{\sqrt{3}}{2} + \frac{4\pi}{3}$

(e) $\int_{-e^2}^{-e} \frac{3}{t}dt = 3\ln|t| \Big|_{-e^2}^{-e} = 3\ln|-e| - 3\ln|-e^2| = 3\ln e - 3\ln(e^2) =$
$= 3 \cdot 1 - 3 \cdot 2 = -3$

7. Find the derivatives of the following functions

(a) $f(x) = \int_{2}^{x} \sin(t^2)dt$
$f'(x) = \sin(x^2)$

(b) $g(x) = \int_{3x}^{5x^2} \sqrt{t}\tan(3t)dt$
$g'(x) = \sqrt{5x^2}\tan(3 \cdot 5x^2) \cdot 10x - \sqrt{3x}\tan(3 \cdot 3x) \cdot 3 = 10\sqrt{5}x^2\tan(15x^2) - 3\sqrt{3x}\tan(9x)$

8. Estimate the value of $\int_{0}^{10} (x^2+6)dx$ using 5 approximating rectangles and

We divide $[0, 10]$ into 5 subintervals of length $\Delta x = \frac{10-0}{5} = 2$, i.e. $[0, 2]$, $[2, 4]$, $[4, 6]$, $[6, 8]$, $[8, 10]$.

(a) left endpoints,
Left endpoints are 0, 2, 4, 6, and 8. So $L_5 = (f(0) + f(2) + f(4) + f(6) + f(8))\Delta x = (6 + 10 + 22 + 42 + 70)2 = 300.$

(b) right endpoints,
Right endpoints are 2, 4, 6, 8, and 10. So $R_5 = (f(2) + f(4) + f(6) + f(8) + f(10))\Delta x = (10 + 22 + 42 + 70 + 106)2 = 500.$

(c) midpoints.
Midpoins are 1, 3, 5, 7, and 9. So So $M_5 = (f(1) + f(3) + f(5) + f(7) + f(9))\Delta x = (7 + 15 + 31 + 55 + 87)2 = 390.$

(d) Evaluate $\int_{0}^{10} (x^2+6)dx$ using the Fundamental Theorem of Calculus.

$$= \left(\frac{1}{3}x^3 + 6x\right)\Big|_{0}^{10} = \left(\frac{1000}{3} + 60\right) - 0 = 393.333333$$

(e) Sketch the graph of $f(x) = x^2 + 6$ and explain the meaning of your answers in (a)-(d).
The answers in (a)-(d) are areas of the shaded regions below.

9. Evaluate the following integrals

(a) $\int x\sin(x^2)dx$

$u = x^2,\ du = 2xdx,\ \frac{du}{2} = xdx$, so

$= \int \frac{1}{2}\sin(u)du = -\frac{1}{2}\cos(u) + c = -\frac{1}{2}\cos(x^2) + c$

(b) $\int_0^{\pi/4} \sec^2 x e^{\tan x} dx$

$u = \tan x,\ du = \sec^2 x dx$,

change the limits of integration: the upper limit becomes $\tan(\pi/4) = 1$, and the lower limit becomes $\tan(0) = 0$.

$= \int_0^1 e^u du = e^1 - e^0 = e - 1$

Another way: evaluate the indefinite integral first,

$\int \sec^2 x e^{\tan x} dx = \int e^u du = e^u + c = e^{\tan x} + c$, then use the old limits:

$\int_0^{\pi/4} \sec^2 x e^{\tan x} dx = e^{\tan(\pi/4)} - e^{\tan(0)} = e^1 - e^0 = e - 1$

(c) $\int 3\cot t dt = 3\int \frac{\cos t}{\sin t} dt$

$u = \sin t,\ du = \cos t dt$,

$= 3\int \frac{1}{u} du = 3\ln|u| + c = 3\ln|\sin t| + c$

(d) $\int \frac{1}{(2-3s)^5} ds$

$u = 2 - 3s,\ du = -3ds,\ -\frac{du}{3} = ds$,

$= -\frac{1}{3}\int \frac{1}{u^5} du = -\frac{1}{3}\int u^{-5} du = -\frac{1}{3} \cdot \frac{u^{-4}}{-4} + c = \frac{u^{-4}}{12} + c = \frac{(2-3s)^{-4}}{12} + c$