MATH 161

FIRST MIDTERM EXAM

October 4, 2001

8:00-9:15 am (75 minutes)

NAME (please print legibly): ____________________

Your U of R ID Number: ____________________

Circle your Professor's name: Johnson Knapp Mueller Pizer

- No calculators are allowed on this exam.
- Please show all your work. You may not receive full credit for a correct answer if there is no work shown.
- Please indicate your final answer CLEARLY!

QUESTION	VALUE	SCORE
1.	8	
2.	9	
3.	10	
4.	28	
5.	14	
6.	14	
7.	17	
TOTAL	100	

1. (8 pts) Consider the two points $(1, -3)$ and $(11, 3)$.

(a) Find the distance between these two points.

$$\text{distance} = \sqrt{(11-1)^2 + (3-(-3))^2} = \sqrt{136} = 2\sqrt{34}$$

ANSWER: $\underline{\sqrt{136} \text{ or } 2\sqrt{34}}$

(b) Find the equation of the line through these two points.

The slope of the line is

$$\frac{\Delta y}{\Delta x} = \frac{3-(-3)}{11-1} = \frac{6}{10} = \frac{3}{5},$$

and a point on the line is $(11, 3)$. So using the point-slope formula gives the equation

$$y - 3 = \frac{3}{5}(x - 11).$$

Two other possible answers are

$$y + 3 = \frac{3}{5}(x - 1)$$

and

$$y = \frac{3}{5}x - \frac{18}{5}.$$

ANSWER: $\underline{y - 3 = \frac{3}{5}(x - 11)}$

2. (9 pts)

(a) If $\ln(a) = 7$ and $\ln(b) = 4$, what is $\ln\left(\frac{ea^5}{\sqrt{b}}\right)$?

$$\ln\left(\frac{ea^5}{\sqrt{b}}\right) = \ln\left(ea^5\right) - \ln\left(\sqrt{b}\right) = \ln\left(ea^5\right) - \ln\left(b^{1/2}\right) = \ln(e) + 5\ln(a) - \frac{1}{2}\ln(b) = 1 + 5(7) - \frac{1}{2}(4) = 34.$$

ANSWER: 34

(b) If $\ln(2x-1) = 3$, what is x?

$$\begin{aligned} \ln(2x-1) &= 3 \\ e^{\ln(2x-1)} &= e^3 \\ 2x - 1 &= e^3 \\ 2x &= e^3 + 1 \\ x &= \frac{1}{2}(e^3+1) \end{aligned}$$

ANSWER: $\frac{1}{2}(e^3+1)$

(c) If $2^{-x} = 4$, what is x?

$$\begin{aligned} 2^{-x} &= 4 \\ \log_2(2^{-x}) &= \log_2(4) \\ -x &= 2 \\ x &= -2 \end{aligned}$$

ANSWER: $x = -2$

3. (10 pts) Let $f(x) = 3x + 2$ and $g(x) = x^2 + 1$. Find

(a) $(f - g)(2)$

Note that $f(2) = 8$ and $g(2) = 5$. Then we have

$$(f - g)(2) = f(2) - g(2) = 8 - 5 = 3.$$

ANSWER (a): <u>3</u>

(b) $(fg)(2) = (f \cdot g)(2)$

$$(fg)(2) = f(2)g(2) = (8)(5) = 40.$$

ANSWER (b): <u>40</u>

(c) $f(g(2)) = (f \circ g)(2)$

First, find $g(2)$, and then plug the answer into $f(x)$.

$$f(g(2)) = f(5) = 3(5) + 2 = 17.$$

ANSWER (c): <u>17</u>

(d) $f^{-1}(g(-1)) = (f^{-1} \circ g)(-1)$

First, note that $g(-1) = 2$, so that

$$f^{-1}(g(-1)) = f^{-1}(2).$$

Now, $f^{-1}(2)$ is the x-value that you have to plug in to get $f(x) = 2$. To evaluate this, we solve

$$\begin{aligned} f(x) &= 2 \\ 3x + 2 &= 2 \\ 3x &= 0 \\ x &= 0. \end{aligned}$$

So $f^{-1}(2) = 0$, and this is the final answer.

ANSWER (d): <u>0</u>

4. (28 pts) Find the following limits. Your answer should be either a real number, or ∞, $-\infty$, or DNE (Does Not Exist).

(a) $\displaystyle\lim_{x\to 3} \frac{x^2+x-6}{x+3}$

Since the function is a rational function and is defined at $x = 3$, it is continuous there. So we can just plug in $x = 3$ to get

$$\lim_{x\to 3} \frac{x^2+x-6}{x+3} = \frac{3^2+3-6}{3+3} = 1.$$

ANSWER (a): 1

(b) $\displaystyle\lim_{x\to 2^-} \frac{x^2+4}{x^2-4}$

As x gets closer and closer to 2, the top of the fraction gets close to 8, and the bottom gets close to 0. Moreover, since x is always less than 2, the denominator of the fraction is always negative. So the entire fraction gets more and more negative, and the limit is $-\infty$.

ANSWER (b): $-\infty$

(c) $\displaystyle\lim_{x\to 2} \frac{x^2+2x-8}{x^2-3x+2}$

$$\lim_{x\to 2} \frac{x^2+2x-8}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-2)(x+4)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+4}{x-1} = \frac{2+4}{2-1} = 6.$$

ANSWER (c): 6

(d) $\displaystyle\lim_{x\to \infty} \frac{3x^2+2}{x^2-x-2}$

$$\lim_{x\to\infty} \frac{3x^2+2}{x^2-x-2} = \lim_{x\to\infty} \frac{3x^2+2}{x^2-x-2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{2}{x^2}}{1-\frac{1}{x}-\frac{2}{x^2}} = \frac{3+0}{1-0-0} = 3.$$

ANSWER (d): 3

(e) $\lim_{x\to\infty} \frac{\sin(x)}{1+x^2}$

Since $-1 \le \sin x \le 1$, we have

$$\frac{-1}{1+x^2} \le \frac{\sin x}{1+x^2} \le \frac{1}{1+x^2}.$$

Hence the limit laws say that we have

$$\lim_{x\to\infty} \frac{-1}{1+x^2} \le \lim_{x\to\infty} \frac{\sin x}{1+x^2} \le \lim_{x\to\infty} \frac{1}{1+x^2}.$$

The largest and smallest of these limits are both equal to zero, so the squeeze theorem says that the middle one also has to be zero.

ANSWER (e): <u>0</u>

(f) $\lim_{x\to -3} \frac{7}{x+3}$

As x gets close to -3, the numerator of the fraction is always 7, and the denominator gets close to 0. If x is larger than -3, then the fraction is always positive, and so the limit FROM THE RIGHT is ∞. If x is smaller than -3, then the fraction is negative, and so the limit FROM THE LEFT is $-\infty$. Since the right and left hand limits are different, the regular limit does not exist.

ANSWER (f): <u>DNE</u>

(g) $\lim_{x\to 1^+} \frac{1-x^2}{|1-x|}$

Since we're only considering values of x which are larger than 1, the expression $1-x$ is negative, and so we have $|1-x| = -(1-x)$. Therefore, we have

$$\lim_{x\to 1^+} \frac{1-x^2}{|1-x|} = \lim_{x\to 1^+} \frac{1-x^2}{-(1-x)} = \lim_{x\to 1^+} \frac{(1+x)(1-x)}{-(1-x)} = \lim_{x\to 1^+} \frac{1+x}{-1} = \frac{1+1}{-1} = -2.$$

ANSWER (g): <u>-2</u>

5. (14 pts) Determine constants a and b so that the function $f(x)$ defined by

$$f(x) = \begin{cases} \cos(\pi x) + a & \text{if } x < 1 \\ b & \text{if } x = 1 \\ \sqrt{6x-2} & \text{if } x > 1 \end{cases}$$

is continuous on $(-\infty, \infty)$.

Since the functions $\cos(\pi x) + a$ and $\sqrt{6x-2}$ are both continuous no matter what the value of a is, the only possible problem is that $f(x)$ might not be continuous at $x = 1$. To make the function continuous at $x = 1$, you need to have

$$\lim_{x \to 1} f(x) = f(1).$$

Since $f(1) = b$, you need to make sure that

$$\lim_{x \to 1^+} f(x) = b \qquad \text{and} \qquad \lim_{x \to 1^-} f(x) = b.$$

So first we need

$$b = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{6x-2} = \sqrt{6(1)-2} = \sqrt{4} = 2.$$

Then we also need

$$b = \lim_{x \to -^-} f(x) = \lim_{x \to 1^-} \cos(\pi x) + a = \cos(\pi(1)) + a = (-1) + a.$$

Since we have $b = 2$, this becomes $2 = (-1) + a$, and so we get $a = 3$.

ANSWER: $a = \underline{3}$

ANSWER: $b = \underline{2}$

6. (14 pts) Sketch the graph of an example of a function $f(x)$ which satisfies the following:

the domain of $f(x)$ is the interval $(0, \infty)$

the range of $f(x)$ is the interval $[4, -\infty)$

$f(2) = 4$

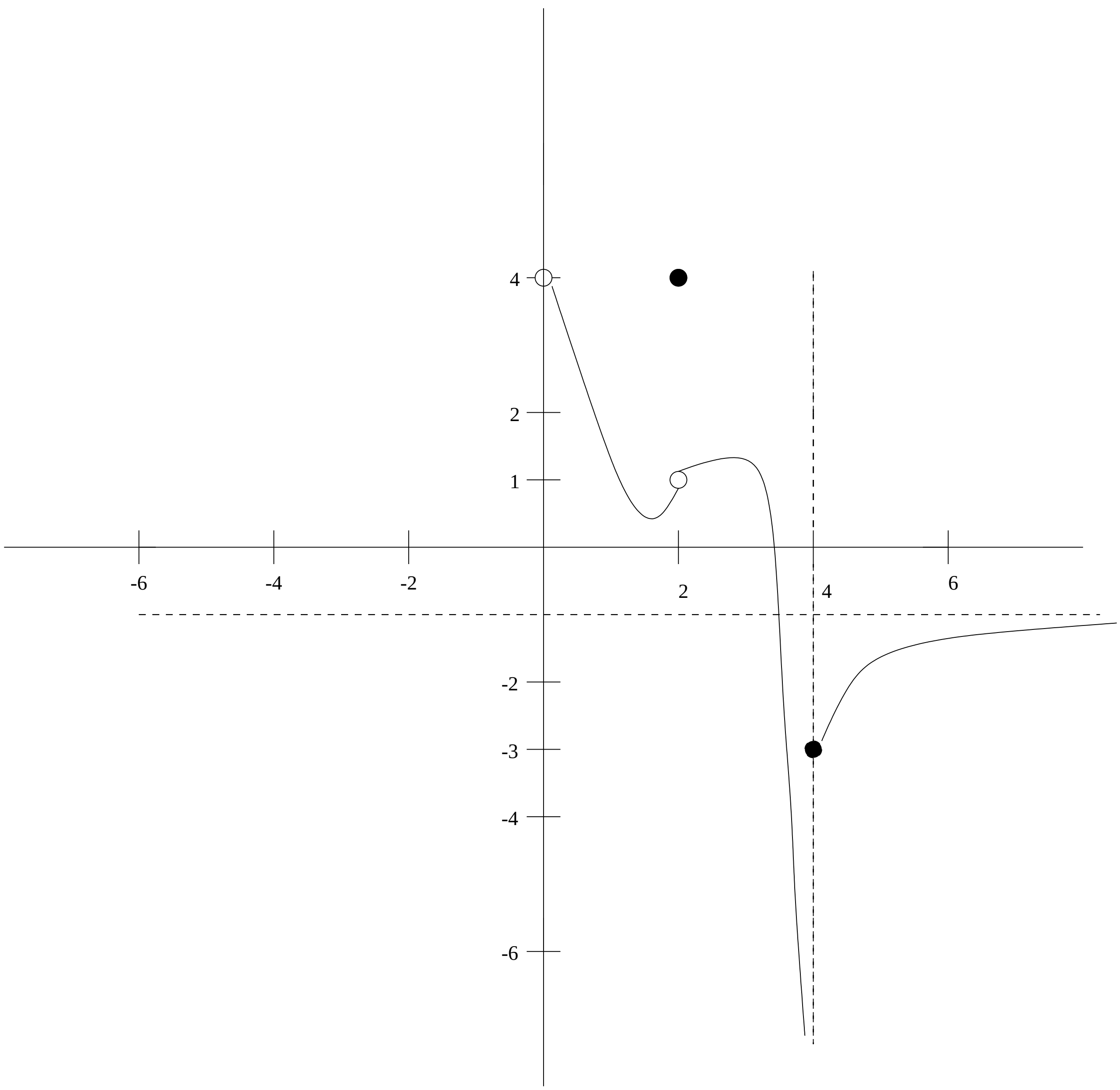
$\lim_{x \to 2} f(x) = 1$

$\lim_{x \to 4^-} f(x) = -\infty$

$\lim_{x \to 4^+} f(x) = -3$

$\lim_{x \to \infty} f(x) = -1,$

One possible function would be the one below.



7. (17 pts) Let $y = \sqrt{3x+1}$.

(a) Find the average rate of change of y (with respect to x) as x changes from 0 to 1.

The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{y(1) - y(0)}{1 - 0} = \frac{2-1}{1-0} = 1.$$

ANSWER: <u>1</u>

(b) Write down a limit which gives the instantaneous rate of change of y (with respect to x) at $x = 1$.

The limit is

$$\lim_{h\to 0} \frac{\sqrt{3(1+h)+1} - 2}{h}.$$

Another possible answer is

$$\lim_{x\to 1} \frac{\sqrt{3x+1} - 2}{x-1}.$$

ANSWER: $\underline{\lim_{h\to 0} \frac{\sqrt{3(1+h)+1} - 2}{h}}$

(c) Evaluate the limit in (b) to find the instantaneous rate of change of y at $x = 1$.

$$\lim_{h\to 0} \frac{\sqrt{3(1+h)+1} - 2}{h} = \lim_{h\to 0} \frac{\sqrt{4+3h} - 2}{h} = \lim_{h\to 0} \frac{\sqrt{4+3h} - 2}{h} \frac{\sqrt{4+3h} + 2}{\sqrt{4+3h} + 2}$$

$$= \lim_{h\to 0} \frac{(4+3h) - 4}{h(\sqrt{4+3h} + 2)} = \lim_{h\to 0} \frac{3h}{h(\sqrt{4+3h} + 2)} = \lim_{h\to 0} \frac{3}{\sqrt{4+3h} + 2} = \frac{3}{\sqrt{4+3(0)} + 2} = \frac{3}{4}$$

ANSWER: $\underline{\frac{3}{4}}$