161s07	Midterm 2	Exam Time: Thursday 4/5, 8:00 - 9:30

Name:	Student No.:

Instructions:

- Answer ALL questions from Section A
- You may use a handwritten sheet of notes. Calculators are NOT permitted.
- Read all questions carefully
- Unless explicitly told otherwise, you should explain all your answers fully.
- Do NOT seperate the pages of your exam.

Problem	Points	Score
A1	12	
A2	16	
A3	4	
A4	12	
A5	10	
A6	10	
A7	10	
A8	10	
A9	12	
A10	4	
Total	100	

Name:

Section A: Answer ALL questions.

Problem A1: [12 pts] In each case, find $\frac{dy}{dx}$.

(**a**) $y = \dfrac{xe^{-x}}{1+x^2}$

Solution:

Using the quotient-rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+x^2)\frac{d}{dx}\left(xe^{-x}\right) - xe^{-x}\frac{d}{dx}\left(1+x^2\right)}{(1+x^2)^2} \\ &= \frac{(1+x^2)(e^{-x} - xe^{-x}) - xe^{-x}(2x)}{(1+x^2)^2} \\ &= e^{-x}\frac{1 - x - x^2 - x^3}{(1+x^2)^2}\end{aligned}$$

(**b**) $y = (\sin x)^x$

Solution:

Use logarithmic differentiation:

$$\ln y = x \ln(\sin x)$$

so

$$\frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + x\frac{\cos x}{\sin x}$$

hence

$$\frac{dy}{dx} = (\sin x)^x \left(\ln(\sin x) + x \cot x\right).$$

(**c**) $xy^3 = 4$

Solution:

Use implicit differentiation:

$$y^3 + 3xy^2\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = -\frac{y^3}{3xy^2} = -\frac{y}{3x}.$$

Name:

Problem A2: [16 pts] Find the following integrals.

(**a**) $\displaystyle\int \frac{3}{1+x^2} - \frac{1}{x^2}\,dx$

Solution:

$$\int \frac{3}{1+x^2} - \frac{2}{x^3}\,dx = 3\arctan(x) - 2\left(\frac{1}{-2}x^{-2}\right) + C = 3\arctan(x) + \frac{1}{x^2} + C$$

(**b**) $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

Solution:

Let $u = \ln x$ then $du = \frac{1}{x}dx$ so

$$\begin{aligned}\int \frac{\sin(\ln x)}{x}\,dx &= \int \sin(u), du \\ &= -\cos(u) + C \\ &= -\cos(\ln x) + C\end{aligned}$$

(**c**) $\displaystyle\int_1^2 x^3 - e^x\,dx$

Solution:

$$\int_0^2 x^3 - e^x\,dx = \left[\frac{1}{4}x^4 - e^x\right]_1^2 = \left(\frac{16}{4} - e^2\right) - (0-1) = 5 - e^2$$

(**d**) $\displaystyle\int_1^4 \frac{1}{1+\sqrt{u}}\,dx$

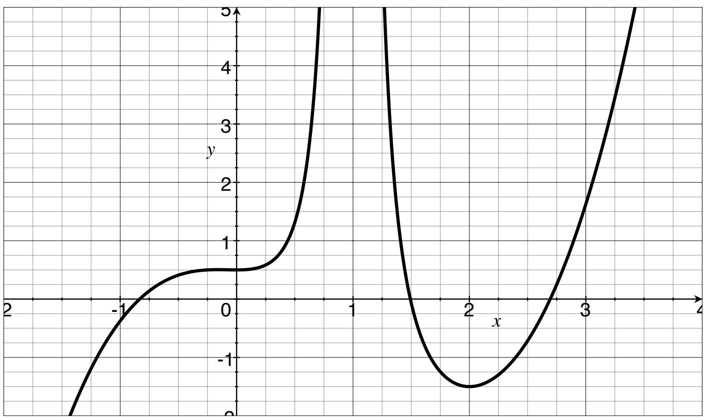
Solution:

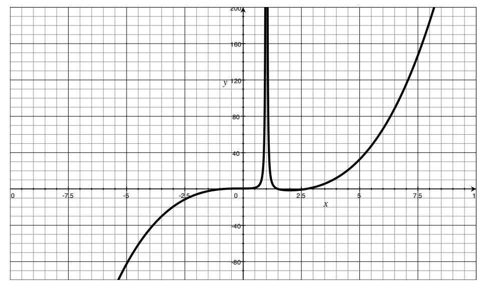
Let $u = \sqrt{x}$ then $du = \frac{1}{2\sqrt{x}}dx$ so $dx = 2udu$. When $x = 1$, $u = \sqrt{1} = 1$. When $x = 4$, $u = \sqrt{4} = 2$. Thus

$$\begin{aligned}\int_{x=1}^4 \frac{1}{1+\sqrt{u}}\,dx &= \int_{u=1}^2 \frac{2u}{1+u}du \\ &= 2\int_1^2 \frac{u+1-1}{1+u}du = 2\int_1^2 1 - \frac{1}{1+u}du \\ &= \Big[u - \ln|1+u|\Big]_1^2 = (2-\ln 3) - (1 - \ln 2) \\ &= 1 - \ln\frac{3}{2}\end{aligned}$$

Name:

Problem A3: [4 pts] Two graphs of a function $f(x)$ are given below on axes with different scales.





(**a**) What are the critical values of $f(x)$?

Solution:

$0, 1, 2$

(**b**) Put the following x-values in order of increasing values of $f'(x)$. $x = -\frac{1}{2}, 0, \frac{3}{2}, 3$.

Solution:

$\frac{3}{2}, 0, -\frac{1}{2}, 3$

(**c**) At which of the following values is $f''(x)$ positive? $x = -1, \frac{1}{2}, 2, 3$.

Solution:

$\frac{1}{2}, 2$

(**d**) Is $\displaystyle\int_2^3 f(x)dx$ positive or negative?

Solution:

Negative

Name:

Problem A4: [12 pts] Find the following limits:

(a) $\lim_{x\to\infty} \frac{2x^3 - 3x^2 + 6}{(x+2)^3}$

Solution:

Divide top and bottom by x^3

$$\lim_{x\to\infty} \frac{2x^3 - 3x^2 + 6}{(x+2)^3} = \lim_{x\to\infty} \frac{2 - 3/x + 6/x^3}{(x + 2/x)^3} = \frac{2 - 0 + 0}{(1+0)^3} = 2.$$

(b) $\lim_{x\to 0} x^{\sin x}$

Solution:

The form of the limit is 0^0 which is indeterminate, so we must take a logarithm and convert to a quotient so we can use L'Hospitals Rule.

$$\begin{aligned}
\lim_{x\to 0} \ln\left(x^{\sin x}\right) &= \lim_{x\to 0} \sin x \ln x \\
&= \lim_{x\to 0} \frac{\ln x}{\csc x} \\
&\overset{L'H}{=} \lim_{x\to 0} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0} \frac{-\sin x \tan x}{x} \\
&\overset{L"H}{=} -\lim_{x\to 0} \frac{\cos x \tan x + \sin x \sec^2 x}{1} = -\frac{0+0}{1} = 0
\end{aligned}$$

This was however the ln of the limit we needed to find, so

$$\lim_{x\to 0} x^{\sin x} = e^0 = 1.$$

Name:

Problem A5: [10 pts] Ann is standing on a muddy field. Three kilometers due east is a road running north-south. Five kilometers down the road (north) is a farm. She can walk over the field at 2 km/h and along the road at 5 km/h. If she walks to the road and then along the road by the fastest route, where does she join the road?

Solution:

Set up a coordinate axis so that the road is the y-axis, Ann is standing at $(-3, 0)$ and the farm is at $(0, 5)$. Let the point at which she joins the road be $P = (0, y)$.

Her distance to P is $\sqrt{9 + y^2}$. The distance from P to the farm is $|5 - y|$.

It doesn't make sense to walk south or further north than the farm, so we can take $0 \le y \le 5$. Thus we need to minimize

$$T(y) = \frac{1}{2}\sqrt{9 + y^2} + \frac{5 - y}{5} \qquad \text{on } 0 \le y \le 5.$$

Now $\frac{dT}{dy} = \frac{y}{2\sqrt{9+y^2}} - \frac{1}{5}$ so $\frac{dT}{dy} = 0$ when $5y = 2\sqrt{9 + y^2}$. Squaring both sides yields $25y^2 = 18 + 2y^2$ so $y^2 = 6$. Since we only need to worry about $0 \le y \le 5$, the only important critical point is therefore $y = \sqrt{6}$.

Now

$$T(0) = \frac{3}{2} + 1 = \frac{5}{2}, \quad T(\sqrt{6}) = \frac{\sqrt{15}}{2} + \frac{5 - \sqrt{15}}{5} \approx 2 + \frac{1}{5}, \quad T(5) = \frac{\sqrt{34}}{2} \approx 3$$

So $y = \sqrt{6}$ is indeed the minimum and she joins the road $\sqrt{6}$ km north of her current position.

Name:

Problem A6: [10 pts] A rocket is fired directly up from a resting position at ground level on a planet where the acceleration due to gravity is 4 m/s^2. For the first 5 seconds its acceleration is given by $a(t) = 6t$ m/s. After 5 seconds, its fuel supply runs out and it accelerates according to gravity. How high does it get?

Solution:

Let $s(t)$ represent its height. Then its acceleration is

$$s''(t) = \begin{cases} 6t, & t < 5 \\ -4, & t \geq 5 \end{cases}$$

Anti-differentiating yields

$$s'(t) = \begin{cases} 3t^2 + c_1 & t < 5 \\ -4t + c_2 & t \geq 5 \end{cases}$$

Now the rocket is fired from rest so $s'(0) = 0$ and $c_1 = 0$. By continuity of velocity, the values at $t = 10$ must match so $3(25)^2 = -4(5) + c_2$, so $c_2 = 75 + 20 = 95$. Thus

$$s'(t) = \begin{cases} t^2 & t < 5 \\ -4t + 95 & t \geq 5 \end{cases}$$

Then

$$s(t) = \begin{cases} t^3 + c_3 & t < 5 \\ -2t^2 + 95t + c_4 & t \geq 5 \end{cases}$$

Again $s(0) = 0$ so $c_3 = 0$. By continuity $125 = -50 + 475 + c_4$ so $c_4 = -300$ and

$$s(t) = \begin{cases} t^3 & t < 5 \\ -2t^2 + 95t - 300 & t \geq 5 \end{cases}$$

Now at the highest point $s'(t) = 0$ so we must solve $-4t + 95 = 0$ so $t = \frac{95}{4}$. The highest point is therefore

$$s(95/4) = -\frac{95^2}{8} + \frac{95^2}{4} - 300 = \frac{95^2}{8} - 300 = \frac{9025 - 2400}{8} = \frac{3625}{8}$$

Name:

Problem A7: [10 pts] A vertical wall 2 m high casts a shadow from the sun onto a flat field. At 6:00 pm, the shadow is 4 m long and growing at a rate of 3 $m/hour$. At what rate is the angle the sun makes with the ground changing at 6:00pm?

Solution:

Let x denote the length of the shadow and θ the angle the sun makes with the ground. Then

$$\tan\theta = \frac{2}{x}$$

so using the chain-rule

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{2}{x^2}\frac{dx}{dt}$$

Now at 6:00pm, $x = 4$, $\tan\theta = \frac{1}{2}$ so $\sec^2\theta = 1 + (1/2)^2 = \frac{5}{4}$ and $\frac{dx}{dt} = 3$. Thus

$$\frac{5}{4}\frac{d\theta}{dt} = -\frac{2}{4^2}\cdot 3$$

and

$$\frac{d\theta}{dt} = -\frac{6}{16}\cdot\frac{4}{5} = \frac{15}{2} \quad rad/hour.$$

Name:

Problem A8: [10 pts] A car has velocity $v(t) = \frac{3}{t+1} - 2$

(**a**) Find the net change in position over the interval $0 \le t \le 2$.

Solution:

$$s(2) - s(0) = \int_0^2 \frac{3}{t+1} - 2\,dt = \Big[3\ln|t+1| - 2t\Big]_0^2 = (3\ln 3 - 6) - (3\ln 1 - 0) = 3\ln 3 - 6$$

(**b**) Find the total distance traveled over the interval $0 \le t \le 2$.

Solution:

$v(t) = 0$ when $\frac{3}{t+1} = 2$ so when $2t + 2 = 3$, i.e. at $t = \frac{1}{2}$. And $v(t) > 0$ when $t < 1/2$, $v(t) < 0$ when $t > 1/2$.

$$\begin{aligned}
\text{Distance} &= \int_0^2 |v(t)|dt = \int_0^{1/2} \frac{3}{t+1} - 2\,dt + \int_{1/2}^2 2 - \frac{3}{t+1}\,dt \\
&= \Big[3\ln|t+1| - 2t\Big]_0^{1/2} + \Big[2t - 3\ln|t+1|\Big]_{1/2}^2 \\
&= (3\ln\frac{3}{2} - 1) - (0 - 0) + (6 - 3\ln 3) - (1 - 3\ln\frac{3}{2}) \\
&= 4 + 6\ln\frac{3}{2} - 3\ln 3 \\
&= 4 + 3\ln 3 - 6\ln 2
\end{aligned}$$

Name:

Problem A9: [12 pts] Consider the function $f(x) = x^3 - 6x^2 + 9x + 3$

(**a**) Find and classify all the critical points of $f(x)$.

Solution:

$$f'(x) = 3x^2 - 12x + 9 = 3(x-3)(x-1)$$

so $f'(x) = 0$ when $x = 1, 3$. There are no points where the derivative is undefined, so $x = 1, 3$ are the only critical points.

$$f'' = 6x - 12$$

Then $f''(1) = -6 < 0$ so $x = 1$ is a local maximum and $f''(3) = 6 > 0$ so $x = 3$ is a local minimum.

(**b**) Find the absolute maximum and absolute minimum values that $f(x)$ takes on the interval $[0, 2]$.

Solution:

We evaluate $f(x)$ at the endpoints and at the critical points inside $[0, 2]$.

$$f(0) = 3, \qquad f(1) = 7, \qquad f(2) = 5$$

Thus the abs max value is 7, the abs min value is 3.

(**c**) On what interval(s) is $f(x)$ concave up? Give your answer in interval notation.

Solution:

$f(x)$ is concave up, where $f''(x) > 0$ so when $6x - 12 > 0$. Therefore $f(x)$ is concave up on $(2, \infty)$.

Name:

Problem A10: [4 pts] A differentiable function $f(x)$ satisfies

$$f(x+y) = f(x) + f(y) + f(x)f(y)$$

for all x, y. If $f(0) = 0$, $f(2) = 5$ and $f'(0) = 3$, find $f'(2)$.

Solution:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{5 + f(h) + 5f(h) - 5}{h} \\ &= \lim_{h\to 0} 6\frac{f(h)}{h} \end{aligned}$$

But

$$3 = f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

so

$$f'(2) = 6 \cdot 3 = 18.$$

Solution:

(Method 2)

$$f(x+2) = f(x) + 5 + 5f(x)$$

so the differentiating both sides yields

$$f'(x+2) = f' + 5f'(x) = 6f'(x).$$

Putting in $x = 0$, we get

$$f'(2) = 6f'(0) = 18$$