161s07	Sample Midterm 2	Exam Time: , 8:00 - 9:30

Name:	Student No.:

Instructions:

- Answer ALL questions from Section A
- You may use a handwritten sheet of notes. Calculators are NOT permitted.
- Read all questions carefully
- Unless explicitly told otherwise, you should explain all your answers fully.
- Do NOT seperate the pages of your exam.

Problem	Points	Score
A1	9	
A2	7	
A3	7	
A4	12	
A5	7	
A6	8	
Total	50	

Name:

Section A: Answer ALL questions.

Problem A1: [9 pts] In each of the following cases, find $\frac{dy}{dx}$. (If a particular method is indicated, you must use that method to receive any credit.)

(**a**) $y = \dfrac{\sin(x^2)}{1+e^x}$

Solution:

Use the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}\big(\sin(x^2)\big)(1+e^x) - \sin(x^2)\frac{d}{dx}\big(1+e^x\big)}{(1+e^x)^2} \\ &= \frac{\cos(x^2)\frac{d}{dx}\big(x^2\big)(1+e^x) - \sin(x^2)e^x}{(1+e^x)^2} \\ &= \frac{2x\cos(x^2)(1+e^x) - \sin(x^2)e^x}{(1+e^x)^2}\end{aligned}$$

(**b**) $4xy^2 = x + 3y$

Solution:

Use implicit differentiation:

$$4y^2 + 4x\frac{d}{dx}\big(y^2\big) = 1 + 3\frac{dy}{dx}$$

so

$$4y^2 + 8xy\frac{dy}{dx} = 1 + 3\frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{4y^2-1}{3+8xy}.$$

(**c**) $y = \dfrac{x^2+1}{(x-1)^4\sqrt{x+2}}$. Use logarithmic differentiation.

Solution:

$$\ln y = \ln\frac{x^2+1}{(x-1)^4\sqrt{x+2)}} = \ln(x^2+1) - 4\ln(x-1) - \frac{1}{2}\ln(x+2)$$

so

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} - \frac{4}{x-1} - \frac{1}{2x+4}$$

and hence

$$\frac{dy}{dx} = \frac{x^2+1}{(x-1)^4\sqrt{x+2}}\left(\frac{2x}{x^2+1} - \frac{4}{x-1} - \frac{1}{2x+4}\right)$$

Name:

Problem A2: [7 pts] A man 2 m tall walks away from a lamp post at 4 m/s. When he is 10 m away from the lamp post, the tip of his shadow is moving at 6 m/s. How tall is the lamp post?

Solution:

Let x be the distance of the man from the lamp post, y be the distance of the tip of his shadow from the lamp post and h be the height of the lamp post. The distance from the man to the tip of his shadow is then $y - x$. By similar triangles

$$\frac{y}{h} = \frac{y - x}{2}.$$

Differentiating implicitly then yields

$$\frac{1}{h}\frac{dy}{dt} = \frac{1}{2}\frac{dy}{dt} - \frac{1}{2}\frac{dx}{dt}.$$

Now when $x = 10$, $\frac{dy}{dt} = 6$ and $\frac{dx}{dt} = 5$ so

$$\frac{6}{h} = 3 - \frac{5}{2} = \frac{1}{2}$$

solving yields $h = 12$. So the lamp post is 12 m tall.

Name:

Problem A3: [7 pts] A car driver is accelerating over rough terrain. Her rate of acceleration depends on the velocity she has already obtained according to the following formula

$$a = \frac{2000 - v^3}{300}.$$

If her current velocity is 10 m/s, estimate her velocity in two seconds time.

Solution:

Since we can't explicitly write the velocity in terms of time, we shall use linear approximation. Set the current time as $t = 0$, so $v(0) = 10$. We must estimate $v(1)$. Linear approximation tells us that

$$v(t) \approx v(0) + v'(0)(t - 0)$$

Now $a = \frac{dv}{dt}$, so

$$v'(t) = \frac{2000 - \big(v(t)\big)^3}{300}.$$

In particular,

$$v'(0) = \frac{2000 - 10^3}{300} = \frac{10}{3}$$

Thus

$$v(2) \approx 10 + \frac{10}{3}(2 - 0) = \frac{50}{3} \quad m/s$$

Name:

Problem A4: [12 pts] Consider the function $f(x) = x^4 - 4x^3 + 10$.

(**a**) Find and classify all the critical points of $f(x)$. Put your answers in the table. (You may not need all the entries)

Solution:

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$. Now this is always defined so the only critical points are when $f'(x) = 0$, i.e. at $x = 0, 3$. Now $f'(-1) = -16 < 0$, $f'(1) = -8 < 0$ and $f'(4) = 64 > 0$ so $f(x)$ is decreasing $x < 0$, decreasing $0 < x < 3$ and increasing $x > 3$.

Critical Point:	0	3			
Type:	Saddle	Min			

(**b**) What are the absolute maximum and absolute minimum values that $f(x)$ can take on the interval $[-1, 4]$? Where do these occur?

Solution:

We must check the critical points and the endpoints. Now $f(-1) = 15$, $f(0) = 10$, $f(3) = -17$, $f(4) = 10$

	Min	Max
Value	-17	15
Occurs at:	$x = 3$	$x = -1$

Name:

(**c**) Where is $f(x)$ concave up? Where is $f(x)$ concave down? Give your answers in interval notation.

Solution:

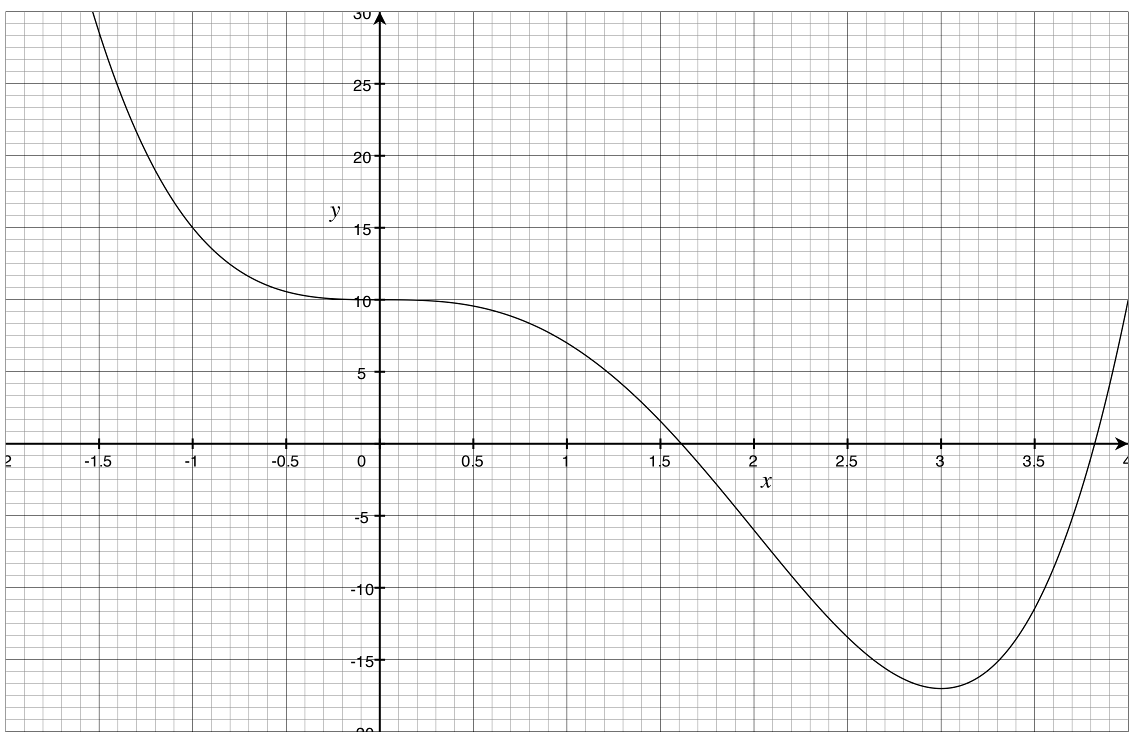
Now $f''(x) = 12x^2 - 24x = 12x(x-2)$ so the inflection points are at $x = 0, 2$. Now $f''(-1) = 36 > 0$, $f''(1) = -12 < 0$ and $f''(3) = 36 > 0$. Thus $f(x)$ is concave up for $x < 0$, concave down $0 < x < 2$ and concave up $x > 2$.

Concave Up	Concave Down
$(0, -\infty) \cup (2, \infty)$	$(0, 2)$

(**d**) Sketch the graph of $y = f(x)$ on the axis below.

Solution:

The additional information we need is that $\lim_{x \to \infty} f(x) = +\infty$ and $\lim_{x \to -\infty} = +\infty$.



Name:

Problem A5: [7 pts] Find $\lim_{x\to 0^+} \big(\sin(x)\ln(x)\big)$

Solution:

The form of this limit is $0 \cdot (-\infty)$ which is indeterminate. To use L'Hopitals rule, we first rewrite as a quotient. As we can use trig identities, it will probably be easier to divide by $\sin x$.

$$\lim_{x\to 0^+} \big(\sin(x)\ln(x)\big) = \lim_{x\to 0^+} \frac{\ln x}{1/\sin x} = \lim_{x\to 0^+} \frac{\ln x}{\csc x}$$
$$\stackrel{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x \cos x}$$

This limit has form 0/0 so we again use L'Hopitals Rule

$$\lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x} \stackrel{L'H}{=} \lim_{x\to 0^+} \frac{-2\sin x \cos x}{\cos x - x \sin x}$$
$$= -\frac{0}{1-0} = 0$$

Name:

Problem A6: [8 pts] You are standing on a straight coastline at the point closest to a lighthouse some way out at sea. You time that it takes 3 minutes for the light beam to make a full rotation. If it takes the light beam 2 seconds to move between you and a point 100 m further down the coast, estimate how far out to sea the lighthouse is.

Solution:

Let h be the distance from the lighthouse to the shore, θ the rotation angle of the light beam and x the distance of the point where the light beam hits the coast from your current position. Then

$$x = h\tan\theta$$

and so

$$\frac{dx}{dt} = h\sec^2\theta\frac{d\theta}{dt}.$$

Now since the beam takes 3 minutes (or 180 seconds) to rotate, we know

$$\frac{d\theta}{dt} = \frac{2\pi}{180} = \frac{\pi}{90}$$

Since it takes 2 seconds for the light beam to travel 100 m from your position, we see that $\frac{dx}{dt} \approx 50\ m/s$ as the light passes you by. Thus

$$50 \approx h\sec^2(0)\frac{\pi}{90}$$

Now $\sec(0) = 1$ so

$$h \approx \frac{450}{\pi}$$