MATH 161

SECOND MIDTERM EXAM

November 6, 2001
8:00-9:15 am (75 minutes)

NAME (please print legibly): ______________________________

Your U of R ID Number: ______________________________

Circle your Professor's name: Johnson Knapp Mueller Pizer

- No calculators are allowed on this exam.
- Please show all your work. You may not receive full credit for a correct answer if there is no work shown.
- Please indicate your final answer CLEARLY!

QUESTION	VALUE	SCORE
1.	20	
2.	20	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
TOTAL	100	

1. (20 pts) Find the following derivatives $y' = \frac{dy}{dx}$. You do not have to simplify your answers.

(a) $y = \dfrac{\cos(x)}{1+\tan(x)}$

$$\begin{aligned} y' &= \frac{(1+\tan x)(\cos x)' - (\cos x)(1+\tan x)'}{(1+\tan x)^2} \\ &= \frac{(1+\tan x)(-\sin x) - (\cos x)(\sec^2 x)}{1+2\tan x+\tan^2 x} \\ &= \frac{-\sin x - \sin x \tan x - \sec x}{1+2\tan x+\tan^2 x}. \end{aligned}$$

ANSWER:______________________________

(b) $y = 3x^5 + \dfrac{6x}{x^2} - \sqrt[3]{x}$

First simplify the formula to

$$y = 3x^5 + \frac{6}{x} - x^{1/3} = 3x^5 + 6x^{-1} - x^{1/3}.$$

Then the derivative is

$$y' = 3(5)x^4 + 6(-1)x^{-2} - \frac{1}{3}x^{-2/3} = 15x^4 - \frac{6}{x^2} - \frac{1}{3}x^{-2/3}.$$

ANSWER:______________________________

(c) $y = e^{x^3-1}$

$$y' = e^{x^3-1}(x^3-1)' = e^{x^3-1}(3x^2) = 3x^2e^{x^3-1}.$$

ANSWER:____________________________

(d) $y = (x^2+1)^5x^{10}(x^4+1)^3$

Use logarithmic differentiation for this problem.

$$\begin{aligned}
y &= (x^2+1)^5x^{10}(x^4+1)^3 \\
\ln y &= \ln\left[(x^2+1)^5x^{10}(x^4+1)^3\right] \\
\ln y &= 5\ln(x^2+1) + 10\ln x + 3\ln(x^4+1) \\
\frac{d}{dx}(\ln y) &= \frac{d}{dx}\Big(5\ln(x^2+1) + 10\ln x + 3\ln(x^4+1)\Big) \\
\frac{1}{y}y' &= 5\frac{(x^2+1)'}{x^2+1} + 10\frac{(x)'}{x} + 3\frac{(x^4+1)'}{x^4+1} \\
\frac{1}{y}y' &= 5\frac{2x}{x^2+1} + 10\frac{1}{x} + 3\frac{4x^3}{x^4+1} \\
\frac{1}{y}y' &= \frac{10x}{x^2+1} + \frac{10}{x} + \frac{12x^3}{x^4+1} \\
y' &= y\left(\frac{10x}{x^2+1} + \frac{10}{x} + \frac{12x^3}{x^4+1}\right) \\
y' &= \left((x^2+1)^5x^{10}(x^4+1)^3\right)\left(\frac{10x}{x^2+1} + \frac{10}{x} + \frac{12x^3}{x^4+1}\right).
\end{aligned}$$

ANSWER:____________________________

(e) $y = x^3\cos(x)$

$$y' = (x^3)'(\cos x) + (x^3)(\cos x)' = 3x^2\cos x + x^3(-\sin x).$$

ANSWER:____________________________

2. (20 pts) Find the following derivatives $y' = \frac{dy}{dx}$. You do not need to simplify your answers.

(a) $y = \sin(\ell n(x^2+1))$

$$\begin{aligned} y' &= \Big(\cos(\ln(x^2+1))\Big)\Big(\ln(x^2+1)\Big)' \\ &= \Big(\cos(\ln(x^2+1)\Big)\left(\frac{1}{x^2+1}\right)\Big(x^2+1\Big)' \\ &= \Big(\cos(\ln(x^2+1)\Big)\left(\frac{1}{x^2+1}\right)(2x). \end{aligned}$$

ANSWER: ______________________________

(b) $y = x^{\ell n(x)}$

Find this derivative by logarithmic differentiation.

$$\begin{aligned} y &= x^{\ln x} \\ \ln y &= \ln\left(x^{\ln x}\right) \\ \ln y &= (\ln x)(\ln x) = (\ln x)^2 \\ \frac{d}{dx}(\ln y) &= \frac{d}{dx}\left((\ln x)^2\right) \\ \frac{1}{y}y' &= 2(\ln x)(\ln x)' \\ \frac{1}{y}y' &= 2(\ln x)\frac{1}{x} \\ y' &= y\left(2(\ln x)\frac{1}{x}\right) \\ y' &= \left(x^{\ln x}\right)\left(2(\ln x)\frac{1}{x}\right). \end{aligned}$$

ANSWER: ______________________________

(c) $y = \sqrt{e^x \cos(x)}$

Note that we have

$$y = (e^x \cos x)^{1/2}.$$

Therefore we have

$$y' = \frac{1}{2}(e^x \cos x)^{-1/2}(e^x \cos x)' = \frac{1}{2}(e^x \cos x)^{-1/2}(e^x \cos x + e^x(-\sin x)).$$

ANSWER: ____________________

(d) $y = \sqrt{x}\, e^x$

$$y' = (\sqrt{x})'(e^x) + (\sqrt{x})(e^x)' = \frac{1}{2}x^{-1/2}e^x + (\sqrt{x})(e^x).$$

ANSWER: ____________________

(e) $\sin(xy) = y^2$

You need to use implicit differentiation on this problem.

$$\begin{aligned}
\sin(xy) &= y^2 \\
\frac{d}{dx}(\sin(xy)) &= \frac{d}{dx}\left(y^2\right) \\
(\cos(xy))\frac{d}{dx}(xy) &= 2y\frac{dy}{dx} \\
(\cos(xy))\left(x\frac{dy}{dx} + (1)y\right) &= 2y\frac{dy}{dx} \\
(x\cos(xy))\frac{dy}{dx} + y\cos(xy) &= 2y\frac{dy}{dx} \\
y\cos(xy) &= 2y\frac{dy}{dx} - (x\cos(xy))\frac{dy}{dx} = (2y - x\cos(xy))\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{y\cos(xy)}{2y - x\cos(xy)}.
\end{aligned}$$

ANSWER: ____________________

3. (10 pts) Find the second derivative $y'' = \frac{d^2y}{dx^2}$ of the following functions.

(a) $y = e^{-x^2}$

First, we have

$$y' = e^{-x^2}(-x^2)' = e^{-x^2}(-2x).$$

Then the second derivative is

$$y'' = e^{-x^2}(-2x)' + \left(e^{-x^2}\right)'(-2x) = e^{-x^2}(-2) + \left(e^{-x^2}(-2x)\right)(-2x) = -2e^{-x^2} + 4x^2e^{-x^2}.$$

ANSWER (a): ______________________

(b) $y = \tan^{-1}(x)$ (another name for $\tan^{-1}(x)$ is arctan (x))

You should know that

$$y' = \frac{1}{1+x^2} = (1+x^2)^{-1}.$$

Then the second derivative is

$$y'' = (-1)(1+x^2)^{-2}(1+x^2)' = \frac{-1}{(1+x^2)^2}(2x) = \frac{-2x}{(1+x^2)^2}.$$

ANSWER (b): ______________________

4. (10 pts) Consider the function y defined implicitly by $y^3 + 3xy^2 + 12x = 32$. Find the equation of the tangent line to y at the point $(1, 2)$.

Since we are given a point on the tangent line (the point $(1, 2)$), we only need to find the slope of the tangent line at this point. To do this, we need to use implicit differentiation.

$$\begin{aligned}
y^3 + 3xy^2 + 12x &= 32 \\
\frac{d}{dx}\left(y^3 + 3xy^2 + 12x\right) &= \frac{d}{dx}(32) \\
3y^2\frac{dy}{dx} + 3x\left(2y\frac{dy}{dx}\right) + 3(1)y^2 + 12 &= 0 \\
3y^2\frac{dy}{dx} + 6xy\frac{dy}{dx} + 3y^2 + 12 &= 0 \\
\left(3y^2 + 6xy\right)\frac{dy}{dx} &= -3y^2 - 12 \\
\frac{dy}{dx} &= \frac{-3y^2 - 12}{3y^2 + 6xy}
\end{aligned}$$

This tells us the value of $\frac{dy}{dx}$ at any point (x, y). To find the slope at the point $(1, 2)$, we just plug these numbers in and get a slope of -1. Therefore the tangent line has slope -1 and contains the point $(1, 2)$. The equation of this line is

$$y - 2 = (-1)(x - 1).$$

You could also put this into $y = mx + b$ form (although you don't have to), and in this case the equation is

$$y = -x + 3.$$

ANSWER: ____________________________

5. (10 pts) The height above the ground of an object is given by

$$s(t) = t^3 - 9t^2 + 15t + 2\,, \quad t \geq 0$$

(a) During what time interval(s) is the object travelling up?

We need to find when the velocity is positive. Since the velocity function is the derivative of the position function, we have

$$v(t) = 3t^2 - 18t + 15.$$

So we need to solve the equation

$$3t^2 - 18t + 15 > 0.$$

After dividing both sides by 3 and factoring, we arrive at the equation

$$(t-1)(t-5) > 0.$$

This will happen when both

$$t - 1 > 0 \qquad \text{and} \qquad t - 5 > 0,$$

and also when both

$$t - 1 < 0 \qquad \text{and} \qquad t - 5 < 0.$$

The first situation will occur whenever we have $t > 5$, and the second whenever $t < 1$. We have to be careful in the second situation, though, because it says in the problem that we're only considering times with $t \geq 0$. So we're only in the second situation when $0 \leq t < 1$. So the object is moving upwards when $0 \leq t < 1$, and also when $t > 5$.

ANSWER:____________________________

(b) Find the total distance traveled by the object over the time interval $t = 0$ to $t = 4$.

First we have to note that the object is moving upwards when $0 \leq t < 1$, and downwards when $1 < t < 4$. Then we have

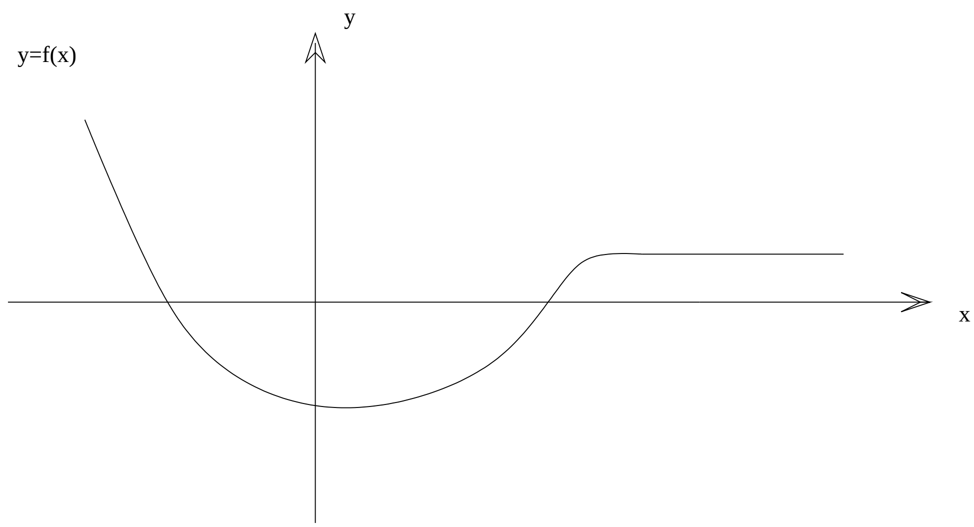
$$\text{Total distance} = \text{Distance upwards} + \text{Distance downwards.}$$

This distance will be

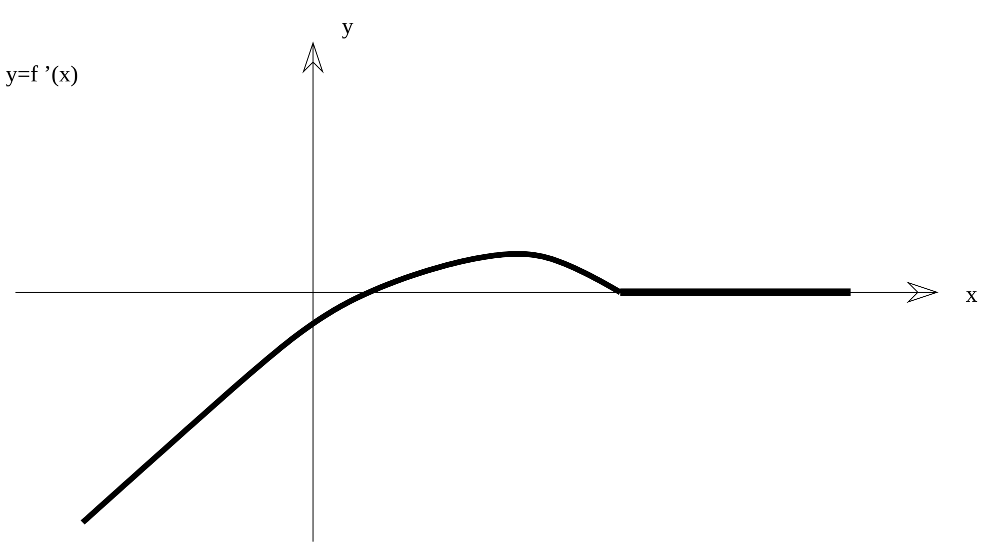
$$|s(1) - s(0)| + |s(4) - s(1)| = |9 - 2| + |(-18) - 9| = |7| + |-27| = 7 + 27 = 34.$$

ANSWER:____________________________

6. (10 pts) Here is the graph of $y = f(x)$. Sketch the graph of $y = f'(x)$.

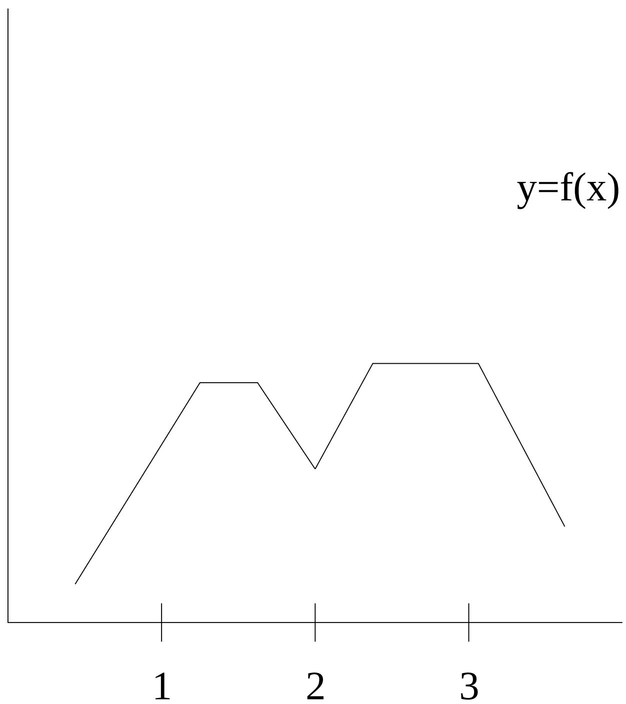


To do this, use the fact that the value of the derivative at a point is the slope of the tangent line to the graph of $f(x)$ at that point. So at the far left, $f'(x)$ should be big and negative, increasing to zero just a bit to the right of the y-axis, where the graph of $f(x)$ has a horizontal tangent line. Then the tangent lines have positive slopes for a while, so $f'(x)$ should be positive. Finally, the graph of $f(x)$ becomes horizontal, and at that point the graph of $f'(x)$ sould become constantly zero. The graph of $f'(x)$ should look something like the one below.

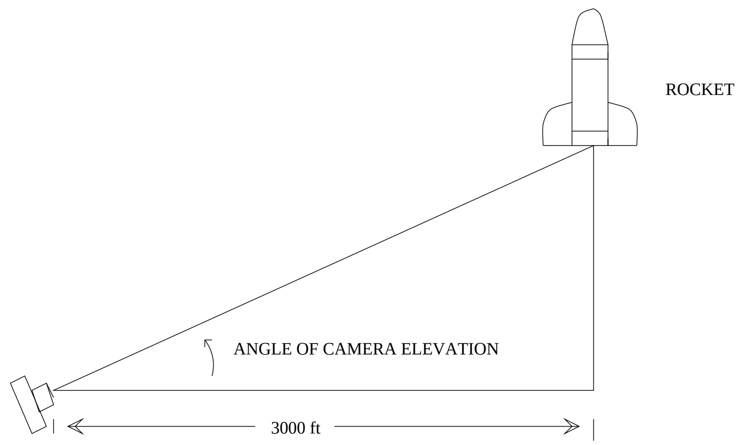


7. (10 pts) Sketch the graph of a continuous function which has a local minimum but is not differentiable at $x = 2$.

To have a local minimum at $x = 2$, the value of $f(x)$ at $x = 2$ must be smaller than the value of $f(x)$ for any value of x close to 2. To make sure that the function is continuous but not differentiable at $x = 2$, there should be either a corner there or else the tangent line at $x = 2$ should be vertical. However, if you try to draw a graph where there is both a local minimum and a vertical tangent line at $x = 2$, you should be able to convince yourself that this isn't possible. So there must be a corner there. A possible solution graph is below.



8. (10 pts) If the rocket shown in the figure is rising at 880 ft/sec when it is 4000 ft above the ground, how fast must the camera elevation angle θ change at that instant to keep the rocket in sight?



First, we need to set up some notation. Let R represent the distance from the rocket to the ground on the graph. Then we have

$$\frac{dR}{dt} = 880 \text{ ft/sec.}$$

Also, we will let the units of θ be radians. We want to relate $\frac{dR}{dt}$ and $\frac{d\theta}{dt}$. To do this, we first need to relate R and θ to each other. Looking at the graph, we can see that we have

$$\tan\theta = \frac{R}{3000}.$$

Now we can take derivatives of both sides with respect to t and get

$$\begin{aligned}
\frac{d}{dt}(\tan\theta) &= \frac{d}{dt}\left(\frac{R}{3000}\right) \\
\left(\sec^2\theta\right)\frac{d\theta}{dt} &= \frac{1}{3000}\frac{dR}{dt} \\
\frac{d\theta}{dt} &= \left(\frac{1}{3000}\right)\left(\frac{dR}{dt}\right)\left(\frac{1}{\sec^2\theta}\right) = \left(\frac{1}{3000}\right)\left(\frac{dR}{dt}\right)\cos^2\theta.
\end{aligned}$$

Now we need to plug in $\frac{dR}{dt}$ and $\cos^2\theta$. We know what $\frac{dR}{dt}$ is, but we need to find the value of $\cos\theta$ when $R = 4000$. To do this, we use the fact that

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}.$$

The adjacent side has length 3000. To find the hypotenuse, we use the Pythagorean theorem:

$$(3000)^2 + (4000)^2 = (\text{hypotenuse})^2.$$

Solving this, we find that the hypotenuse has a length of 5000 feet, and that we have

$$\cos\theta = \frac{3}{5} \qquad \text{and} \qquad \cos^2\theta = \frac{9}{25}.$$

Therefore, the final answer is

$$\frac{d\theta}{dt} = \left(\frac{1}{3000}\right)(880)\left(\frac{9}{25}\right) = \frac{66}{625}\ \text{rad/sec.}$$

ANSWER: ______________________