MATH 161

FINAL EXAM

December 16, 2001
4:00-7:00 pm

NAME (please print legibly): ____
Your U of R ID Number: ____
Circle your Professor's name: Johnson Knapp Mueller Pizer

- No calculators are allowed on this exam.
- Please show all your work. You may not receive full credit for a correct answer if there is no work shown.
- Please indicate your final answer CLEARLY!

PART I

QUESTION	VALUE	SCORE
1.	15	
2.	20	
3.	10	
4.	15	
5.	20	
6.	20	
TOTAL	100	

PART II

QUESTION	VALUE	SCORE
7.	20	
8.	15	
9.	15	
10.	30	
11.	14	
12.	6	
TOTAL	100	

PART I

1. (15 pts) Find the following limits. Write DNE if the limit does not exist.

(a) $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 5x + 6}$

Solution: Note that the limit is of the form 0/0. Using L'Hospital's rule, we get

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{2x + 1}{2x - 5} = \frac{5}{-1} = -5.$$

ANSWER (a):____________________________

(b) $\lim_{x \to 0} \dfrac{e^x - x - 1}{x^2}$

Solution: This time, we use L'Hosiptal's rule twice, to get

$$\lim_{x \to 0} \frac{e^x - x - 1}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.$$

ANSWER (b):____________________________

(c) $\lim_{x \to 4} f(x)$ where $f(x) = \begin{cases} \frac{1}{2}x + 7 & \text{if } x < 4 \\ 10 & \text{if } x = 4 \\ x^2 - 7 & \text{if } x > 4 \end{cases}$

Solution: First, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{1}{2}x + 7 = 9.$$

We also have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} x^2 - 7 = 9.$$

Since the limits from the left and right are equal to each other, the two-sided limit exists and is also equal to 9.

ANSWER (c):____________________________

2. (20 pts) Find the following derivatives $y' = \frac{dy}{dx}$. You do not have to simplify your answers.

(a) $y = (3x^2 + 2x + 1)\ln(x)$

Solution: Using the product rule, we get

$$(3x^2 + 2x + 1)(\ln x)' + (3x^2 + 2x + 1)'(\ln x) = (3x^2 + 2x + 1)\left(\frac{1}{x}\right) + (6x + 2)(\ln x).$$

ANSWER (a): ______________________________

(b) $y = \frac{\sin^{-1}(x)}{e^{4x}}$

Solution: Using the quotient rule, we get

$$y' = \frac{e^{4x}(\sin^{-1}(x))' - (\sin^{-1}(x))\left(e^{4x}\right)'}{\left(e^{4x}\right)^2} = \frac{e^{4x}\frac{1}{\sqrt{1-x^2}} - (\sin^{-1}(x))(4e^{4x})}{e^{8x}}.$$

ANSWER (b): ______________________________

(c) $x^2 + 2xy + y^2 = 7$

Solution: For this one, we need to use implicit differentiation.

$$\begin{aligned}
\frac{d}{dx}\left(x^2 + 2xy + y^2\right) &= \frac{d}{dx}(7) \\
2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\
(2x + 2y)\frac{dy}{dx} &= -(2x + 2y) \\
\frac{dy}{dx} &= -1.
\end{aligned}$$

ANSWER (c): ______________________________

(d) $y = x^{\sin(x)}\ , \quad x > 0$

Solution: This requires logarithmic differentiation.

$$\begin{aligned}
\ln y &= \ln\left(x^{\sin(x)}\right) = (\sin x)(\ln x) \\
\frac{d}{dx}(\ln y) &= \frac{d}{dx}\Big((\sin x)(\ln x)\Big) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{\sin x}{x} + (\cos x)(\ln x) \\
\frac{dy}{dx} &= y\left(\frac{\sin x}{x} + (\cos x)(\ln x)\right) \\
\frac{dy}{dx} &= x^{\sin(x)}\left(\frac{\sin x}{x} + (\cos x)(\ln x)\right).
\end{aligned}$$

ANSWER (d):____

3. (10 pts) Use the following table of values for f, g, f', and g' to answer this question.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	4	3	-2	4
1	-2	-1	4	2
2	7	0	17	-1
3	1	2	13	-3

(a) If $h(x) = f(g(x))$, find $h'(3)$.

Solution: We have $h'(x) = f'(g(x))g'(x)$, so that

$$h'(3) = f'(g(3))g'(3) = f'(2)(-3) = (17)(-3) = -51.$$

ANSWER (a): ____________________

(b) If $k(x) = e^{g(x)}$, find $k'(2)$.

Solution: We have $k'(x) = e^{g(x)}g'(x)$, so that

$$k'(2) = e^{g(2)}g'(2) = e^{0}(-1) = (1)(-1) = -1.$$

ANSWER (b): ____________________

4. (15 pts) The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

Solution: Use the formula $A = \frac{1}{2}bh$, where A is the area, b is the length of the base, and h is the altitude. Taking derivatives of both sides with respect to time, we get

$$\begin{aligned} \frac{d}{dt}(A) &= \frac{d}{dt}\left(\frac{1}{2}bh\right) \\ \frac{dA}{dt} &= \frac{1}{2}\frac{db}{dt}h + \frac{1}{2}b\frac{dh}{dt}. \end{aligned}$$

Now note that if $h = 10$ and $A = 100$, we can solve for b, finding that $b = 20$. We can then plug all the given information into the above equation and get

$$\begin{aligned} 2 &= \frac{1}{2}\frac{db}{dt}(10) + \frac{1}{2}(20)(1) \\ 2 &= 5\frac{db}{dt} + 10 \\ \frac{db}{dt} &= \frac{-8}{5}. \end{aligned}$$

So the length of the base is changing at a rate of -1.6 cm/min.

ANSWER: ______________________

5. (20 pts)

(a) State the limit definition of the derivative of a function $f(x)$.

Solution: There are two possible answers,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{and} \qquad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

(b) Use your definition in part (a) to find the derivative of $f(x) = 3 - x^2$. Note: you must show all necessary work to evaluate your limit. Just writing down $-2x$ is worth 0 points.

Solution: I'll use the first definition to solve this part. We have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(3 - (x+h)^2) - (3 - x^2)}{h} \\ &= \lim_{h \to 0} \frac{(3 - x^2 - 2xh - h^2) - (3 - x^2)}{h} \\ &= \lim_{h \to 0} \frac{-2xh - h^2}{h} \\ &= \lim_{h \to 0} -2x - h \\ &= -2x. \end{aligned}$$

6. (20 pts) Let $f(x) = \sqrt{x}$.

(a) Find the equation of the tangent line to $f(x)$ at $x = 25$.

Solution: If $x = 25$, then $f(x) = 5$. Also, using the fact that $f'(x) = \frac{1}{2}x^{-1/2}$, we find that $f'(25) = \frac{1}{10}$. Now, since $(25, 5)$ is a point on the tangent line, and the slope is $\frac{1}{10}$, the equation of the tangent line is

$$y - 5 = \frac{1}{10}(x - 25) \qquad \text{or, to write it in } y = mx + b \text{ form,} \qquad y = \frac{1}{10}x + \frac{25}{10}.$$

ANSWER (a): ____________________

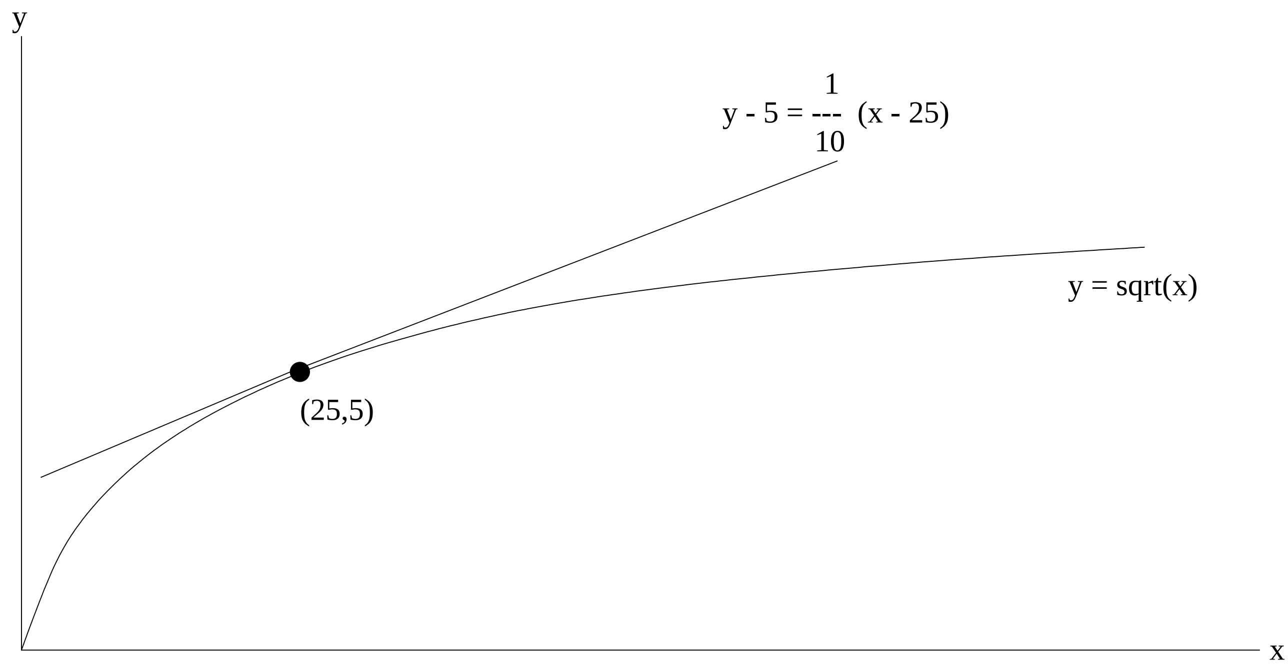
(b) Use linear approximation (i.e. use the equation of the tangent line from part (a)) to approximate $\sqrt{25.5}$.

Solution: For this we simply plug $x = 25.5$ into the equation we found in part (a), getting $y = 5.05$ as the answer.

ANSWER (b): ____________________

(c) Is your approximation too big or too small? State which <u>and</u> draw a picture illustrating why.

Solution: From the drawing below (which is not to scale), we can see that the tangent line is above the graph of $y = \sqrt{x}$ (i.e. the graph is concave down), and so the approximation is too big.



PART II

7. (20 pts) Let $f(x) = x + \frac{1}{x}$. Fill in your answers below. Write "none" if the requested item doesn't exist for this function.

(a) Find the domain of $f(x)$.

Solution: The domain is all values of x where the function is defined. This is all values of x **except** $x = 0$.

domain ____________________

(b) Find all vertical asymptotes of $f(x)$.

Solution: If we rewrite the equation as $f(x) = \frac{x^2+1}{x}$, the vertical asymptotes are at all places where the denominator is undefined. so the only vertical asymptote is the line $x = 0$.

vertical asymptotes ____________________

(c) Find the horizontal asymptotes of $f(x)$.

Solution: We have

$$\lim_{x\to\infty} f(x) = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = -\infty.$$

Since neither of these limits is a number, this function has no horizontal asymptotes.

horizontal asymptotes ____________________

(d) Find the intervals on which $f(x)$ is increasing and the intervals in which $f(x)$ is decreasing.

Solution: The function $f(x)$ is increasing when $f'(x) > 0$, and is decreasing when $f'(x) < 0$. Since we have

$$f'(x) = 1 - \frac{1}{x^2},$$

we can solve and find that $f(x)$ is increasing when

$$1 - \frac{1}{x^2} > 0 \qquad \text{i.e. when} \qquad x^2 > 1.$$

This happens when either $x > 1$ or $x < -1$. Similarly, $f(x)$ is decreasing when $x^2 < 1$, i.e. when $-1 < x < 1$.

increasing ______________________________

decreasing ______________________________

(e) Find the points x at which $f(x)$ has a local maximum and minimum.

Solution: The function $f(x)$ has a local maximum when it changes from increasing to decreasing. This happens at $x = -1$. Similarly, $f(x)$ has a local minimum when it changes from decreasing to increasing. This happens at $x = 1$.

local maximum ______________________________

local minimum ______________________________

(f) Find the values of x at which $f(x)$ has an inflection point and the intervals on which $f(x)$ is concave up and concave down.

Solution: A point is an inflection point if $f(x)$ is defined there and also changes concavity there. In order for this to happen we must have either $f''(x) = 0$ or $f''(x)$ undefined. However, we have

$$f''(x) = \frac{-2}{x^3},$$

which is never equal to zero. Also, the only place where $f''(x)$ is undefined is at $x = 0$, but this is not an inflection point since $f(x)$ is not defined there either. So $f(x)$ has no inflection points. The function $f(x)$ is concave up when we have $f''(x) > 0$, which happens when $x < 0$, and $f(x)$ is concave down when we have $f''(x) < 0$, which happens when $x > 0$.

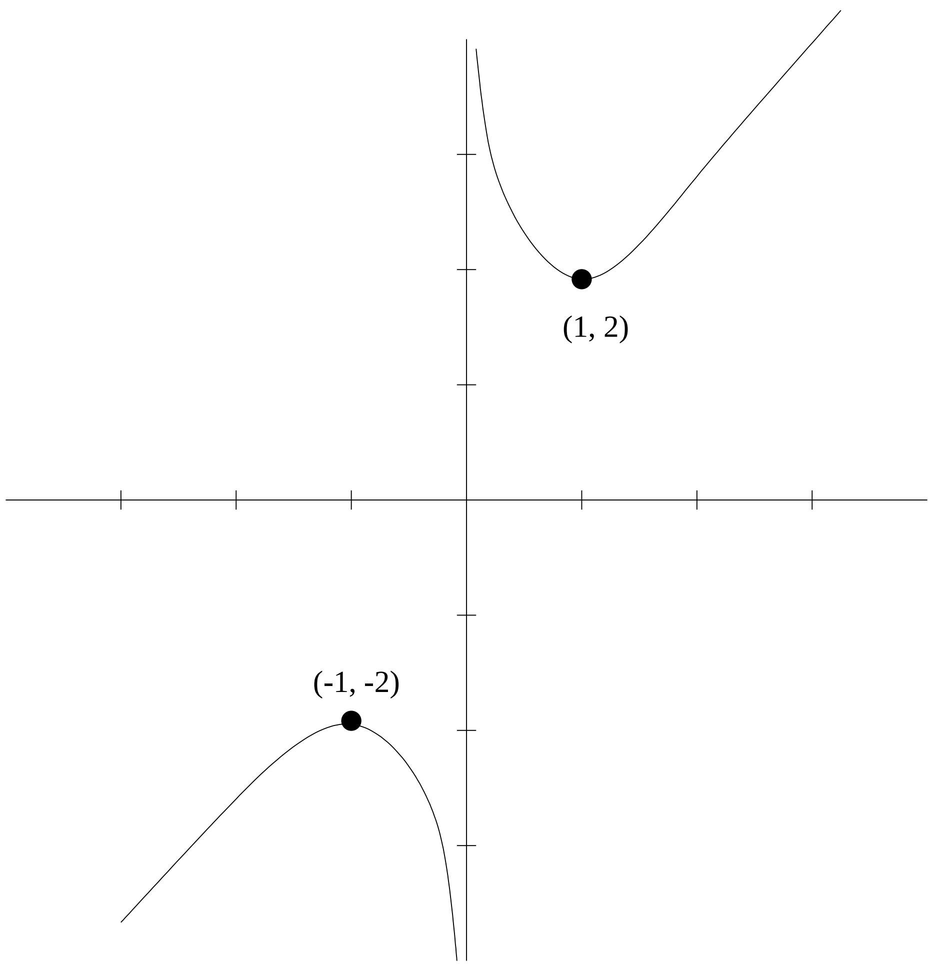
inflection points ____________________

concave up ____________________

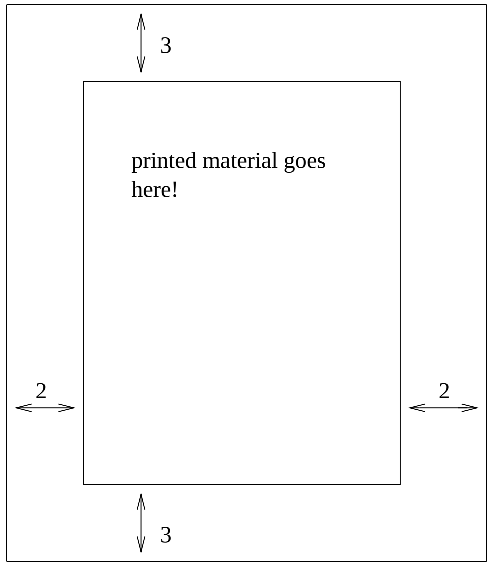
concave down ____________________

(g) Sketch the graph of $f(x)$.

Solution: Using the information from parts (a)-(f) of this problem, the graph of $f(x)$ must look like the one below.



8. (15 pts) The top and bottom margins of a rectangular poster are 3 in. and the side margins are each 2 in. If the area of printed material on the poster is fixed at 600 in^2, find the dimension of the poster with the smallest area.



Solution: Write h for the height of the area of printed material, and w for the width of this area. Then the height of the entire poster is $h+6$ and the width of the entire poster is $w+4$. Then we need to minimize the function $A = (h+6)(w+4)$, subject to the condition $hw = 600$. Now, A involves two variables, so we must solve for one in terms of the other. This is easy, though, since we have $hw = 600$, and so $h = 600/w$. Therefore, the function we have to minimize is

$$A(w) = \left(\frac{600}{w} + 6\right)(w+4) = 624 + 6w + \frac{2400}{w}.$$

The domain of this function will be $w > 0$. First we have to find the critical points by setting $A'(w) = 0$. We have

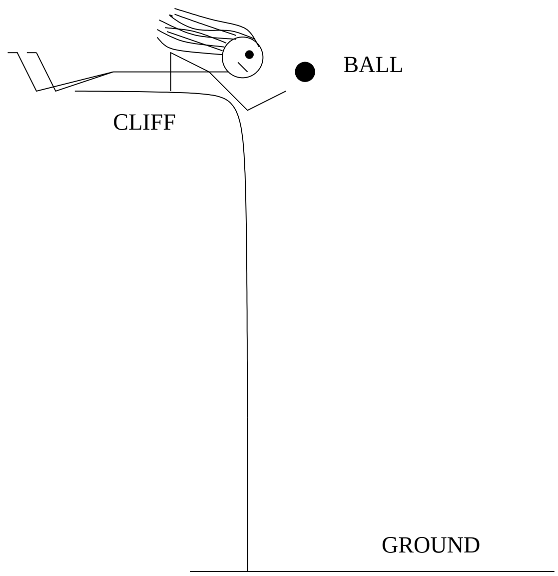
$$A'(w) = 6 - \frac{2400}{w^2}.$$

Setting this equal to 0 and solving for w yields $w = 20$. Noting that we have

$$A''(w) = \frac{4800}{w^3},$$

we see by the second derivative test that this really is a local minimum. Moreover, this has to also be a global minimum. If you think about what the graph of $A(w)$ has to look like, you'll see that if we could have $A(w) < A(20)$ for some value of w, then the graph would have to have a local maximum somewhere. But this cannot happen because there's only one critical point and it's a minimum. Therefore, the poster with the smallest area must have a width of $w + 4 = 24$cm and a height of $h + 6 = 36$cm.

9. (15 pts) If a ball is thrown straight up at a speed of 16 ft/sec from the edge of a cliff 320 ft. above ground level, when will the ball hit the ground? For this problem you can assume gravity is -32 ft/sec^2.



Solution: The problem tells us that we have $s(0) = 320$, $v(0) = 16$ and $a(t) = -32$. First we will find the velocity of the ball as a function of time. We have

$$v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C.$$

To solve for C, we plug in $t = 0$ and $v(0) = 16$. Solving for C gives $C = 16$. So the velocity function for the ball is $v(t) = -32t + 16$. Now we find the position function for the ball. We have

$$s(t) = \int v(t)\,dt = \int -32t + 16\,dt = -16t^2 + 16t + C.$$

To find C, we plug in $t = 0$ and $s(0) = 320$. After solving for C, we find that the position function is $s(t) = -16t^2 + 16t + 320$. Finally, we need to find the time t when the ball hits the ground. To do this, we need to set $s(t) = 0$ and solve for t.

$$\begin{aligned} s(t) &= 0 \\ -16t^2 + 16t + 320 &= 0 \\ 16t^2 - 16t - 320 &= 0 \\ t^2 - t - 20 &= 0 \\ (t-5)(t+4) &= 0 \end{aligned}$$

So the two solutions are $t = 5$ and $t = -4$. We ignore the second solution since we're only interested in times after the ball is thrown. So the ball hits the ground after 5 seconds.

10. (30 pts) Evaluate the following definite integrals.

(a) $\displaystyle\int_1^4 x^3 - \frac{1}{\sqrt{x}}\,dx$

Solution:

$$\int_1^4 x^3 - \frac{1}{\sqrt{x}}\,dx = \int_1^4 x^3 - x^{-1/2}\,dx = \left[\frac{1}{4}x^4 - 2x^{1/2}\right]_1^4 = (64-4) - \left(\frac{1}{4} - 2\right) = 61\frac{3}{4}.$$

ANSWER (a): ____________________

(b) $\displaystyle\int_0^1 \frac{e^x}{1+2e^x}dx$

Solution: We do this by a substitution. Let $u = 1 + 2e^x$. Then we have $du = 2e^x dx$, and so $e^x dx = \frac{1}{2}du$. Therefore, after changing the limits of integration, we have

$$\int_0^1 \frac{e^x}{1+2e^x}dx = \int_3^{1+2e} \frac{1}{2}\cdot\frac{1}{u}du = \left[\frac{1}{2}\ln|u|\right]_3^{1+2e} = \frac{1}{2}\left(\ln(1+2e) - \ln 3\right).$$

ANSWER (b): ____________________

(c) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

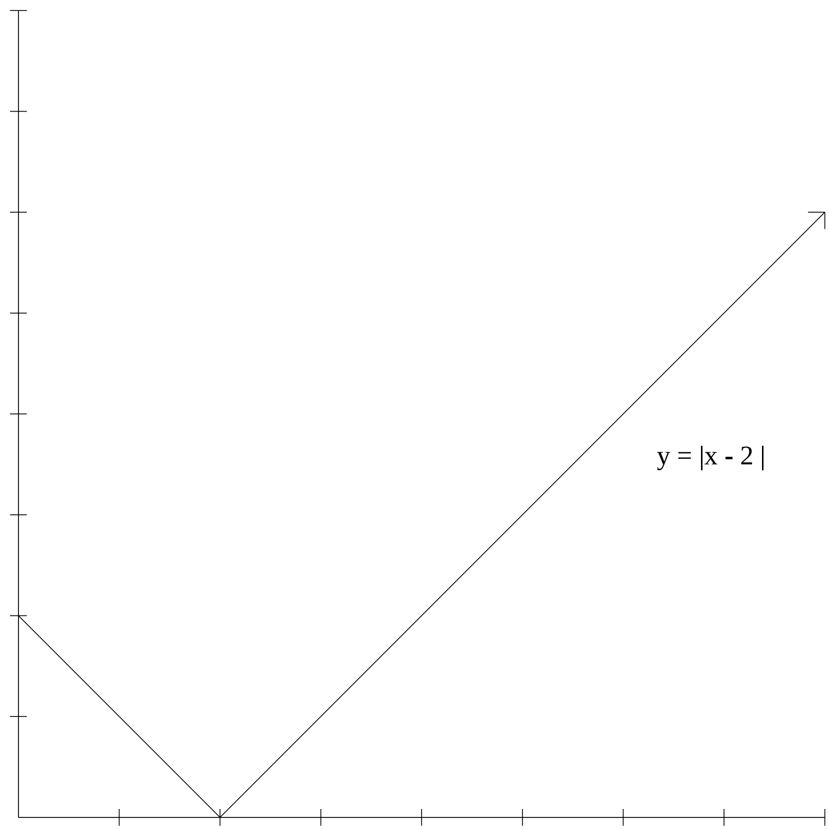
Solution: Since $\dfrac{1}{\sqrt{1-x^2}}$ is the derivative of $\sin^{-1} x$, we have

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1}(x)\right]_0^1 = \left(\sin^{-1}(1) - \sin^{-1}(0)\right) = \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}.$$

ANSWER (c): ____________________

(d) $\int_0^8 |x-2|dx$

Solution: If you graph the function $f(x) = |x-2|$, you'll find that it looks like the diagram below. The left triangle has a base of length 2 and a height of length 2, so its area is $\frac{1}{2}(2)(2) = 2$. The triangle on the right has a base of length 6 and a height of length 6, so its area is $\frac{1}{2}(6)(6) = 18$. The total area under the curve is $2 + 18 = 20$. So the value of the integral is 20.



ANSWER (d): ____________________

(e) $\displaystyle\int_0^{\pi/2} \sin^4(x)\cos(x)dx$

Solution: Make the substitution $u = \sin x$ and $du = \cos x\, dx$. Then after changing the limits, we get

$$\int_0^{\pi/2} \sin^4(x)\cos(x)dx = \int_0^1 u^4\, du = \left[\frac{1}{5}u^5\right]_0^1 = \left(\frac{1}{5} - 0\right) = \frac{1}{5}.$$

ANSWER (e): ______________________

11. (14 pts) Decide whether the following are true (T) or false (F) and mark accordingly.

. (a) If $f'(c) = 0$, then $f(x)$ must have a local maximum or a local minimum
. at $x = c$.

Solution: FALSE: The function might just have a horizontal tangent without having a local minimum or maximum. For example, look at the function $f(x) = x^3$ at $x = 0$.

. (b) $f(x) = |x|$ is continuous on $(-\infty, \infty)$.

Solution: TRUE: If you graph the function, it has no breaks, skips or jumps.

. (c) If $f(x)$ is continuous, then $\int_a^b f(x)dx + \int_b^a f(x)dx = 0$.

Solution: TRUE: If $f(x)$ is continuous, then we have $\int_b^a f(x)dx = -\int_a^b f(x)dx$. Therefore, we have

$$\int_a^b f(x)dx + \int_b^a f(x)dx = \int_a^b f(x)dx - \int_a^b f(x)dx = 0.$$

. (d) $f(x) = x^3 + 3x + 1$ does not have a positive root.

Solution: TRUE: If x is a positive number, then x^3, $3x$ and 1 are all positive numbers. You cannot add up three positive numbers and get zero as an answer.

. (e) If $g'(a)$ exists, then $\lim_{x \to a} g(x) = g(a)$.

Solution: TRUE: The conclusion says that the function $g(x)$ is continuous at a. We saw in class that if a function is differentiable at a point then the function must also be continuous at that point.

. (f) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \frac{\sin(t)}{t}\,dt = \frac{2\sin(x^2)}{x}$.

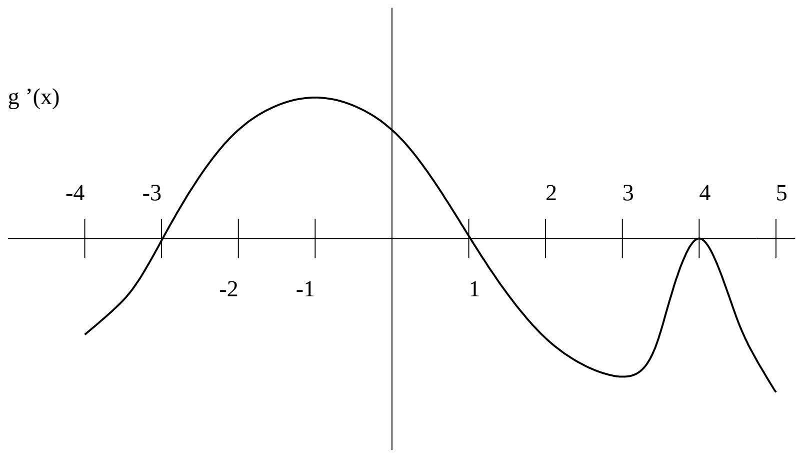
Solution: TRUE: If we write $F(x) = \displaystyle\int_0^{x} \frac{\sin(t)}{t}\,dt$, then we have $F(x^2) = \displaystyle\int_0^{x^2} \frac{\sin(t)}{t}\,dt$. Therefore we can use the chain rule to obtain

$$\begin{aligned}
\frac{d}{dx}\int_0^{x^2} \frac{\sin(t)}{t}\,dt &= \frac{d}{dx}F(x^2) \\
&= F'(x^2)\cdot(2x) \\
&= \left(\frac{\sin\left(x^2\right)}{x^2}\right)(2x) \\
&= \frac{2\sin(x^2)}{x}.
\end{aligned}$$

. (g) $f(x) = x^3$ is concave up on $(-\infty, \infty)$.

Solution: FALSE: The concavity of $f(x)$ is determined by the sign of $f''(x)$. For this function, we have $f''(x) = 6x$. When $x < 0$, we have $f''(x) < 0$, and so $f(x)$ is concave down on the interval $(-\infty, 0)$.

12. (6 pts) The graph of the derivative of $g(x)$ is given below. Using this answer the following questions.



(a) $g(x)$ is increasing on the interval(s) ____________________

(b) $g(x)$ has a local maximum at ____________________

(c) $g(x)$ is concave up on the interval(s) ____________________

Solutions:

(a) The function $g(x)$ is increasing whenever $g'(x)$ is positive. This happens on the interval $-3 < x < 1$.

(b) The function $g(x)$ has a local maximum at the points where $g(x)$ changes from increasing to decreasing. In other words, $g'(x)$ changes from positive to negative. This only occurs at the point $x = 1$.

(c) The function $g(x)$ is concave up whenever the second derivative $g''(x)$ is positive. This happens whenever the first derivative $g'(x)$ is increasing. So $g(x)$ is convave up on the intervals $-4 < x < -1$ and $3 < x < 4$.