

Name:	Student No.:
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Instructions:

- Answer ALL questions from Section A
- You may use a handwritten sheet of notes. Calculators are NOT permitted.
- Read all questions carefully
- Unless explicitly told otherwise, you should explain all your answers fully.
- Do NOT separate the pages of your exam.

Problem	Points	Score
A1	6	<input type="text"/>
A2	4	<input type="text"/>
A3	6	<input type="text"/>
A4	5	<input type="text"/>
A5	9	<input type="text"/>
A6	6	<input type="text"/>
A7	5	<input type="text"/>
A8	9	<input type="text"/>
Total	50	<input type="text"/>

Name:

Section A: Answer ALL questions.

Problem A1: [6 pts]

(a) Solve $3^{-x} = 9^{3+x}$ for x .

Solution:

$9 = 3^2$ so the equation becomes $3^{-x} = 3^{6+2x}$. Using the fact that the exponential functions are one-to-one, this implies that $-x = 6 + 2x$ so $x = -2$

(b) Describe the following set as an interval or union of intervals: $\{x \mid |x - 3| > |x - 1|\}$

Solution:

$|x - 3|$ represents the distance of x from 3. $|x - 1|$ is the distance of x from 1. The set is therefore the set of points strictly closer to 1 than to 3, i.e. everything to the left of 2. The set is the interval $(-\infty, 2)$.

(c) Simplify the following. Your answer should be an integer: $\log_3 4 - 2 \log_3 6$

Solution:

Using the log laws

$$\log_3 4 - 2 \log_3 6 = \log_3 \frac{4}{6^2} = \log_3 \frac{1}{9} = -2$$

(There are other equally valid ways to do this)

Name:

Problem A2: [4 pts] Suppose $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$. Find

(a) The domain of $f \circ g$

Solution:

The domain of f is $[0, \infty)$ and the domain of g is $(-\infty, \infty)$. Thus the domain of $f \circ g$ consists of points x in $(-\infty, \infty)$ such that $x^2 - 1 \geq 0$, i.e. $(-\infty, -1] \cup [1, \infty)$.

(b) The domain of $g \circ f$

Solution:

Everything in the range of f is in the domain of g , Thus the domain of $g \circ f$ is the same as the domain of f , i.e. $[0, \infty)$.

Name:

Problem A3: [6 pts] The population of a mutant breed of amoeba is initially 100 and triples every 2 hours.

(a) The population at time t (in hours) is can be described as

$$p(t) = p_0 e^{kt}$$

Find p_0 and k .

Solution:

$p_0 = p(0) = 100$ is the original population. Now after every 2 hours the population must triple so $e^{2k} = 3$, so $k = \frac{1}{2} \ln 3$.

(b) How long does it take for the population to reach 1000?

Solution:

$p(t) = 100e^{\frac{t}{2} \ln 3} = 1000$ when $\frac{t}{2} \ln 3 = \ln 10$. So at time $t = 2 \frac{\ln 10}{\ln 3}$.

Name:

Problem A4: [5 pts] For what values of c is the function

$$f(x) = \begin{cases} x^2 + c^2, & x \geq 2 \\ 2cx, & x < 2 \end{cases}$$

continuous everywhere?

Solution:

The only possible spot for discontinuity is at $x = 2$. Now $f(2) = 4 + c^2$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2cx) = 4c$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 + c^2) = 4 + c^2$$

so the function is continuous everywhere if $4c = c^2 + 4$. This means we need c to solve the quadratic equation

$$c^2 - 2c + 4 = (c - 2)^2 = 0$$

which has solution $c = 2$.

Name:

Problem A5: [9 pts] Find the following limits. If it doesn't exist say DNE. If it is infinite say whether its $+\infty$ or $-\infty$. Carefully explain your reasoning.

(a) $\lim_{x \rightarrow \infty} \frac{x^3 - 3}{(2x + \sqrt{x})^3}$

Solution:

Divide top and bottom by x^3 . Thus

$$\lim_{x \rightarrow \infty} \frac{x^3 - 3}{(2x + \sqrt{x})^3} = \lim_{x \rightarrow \infty} \frac{1 - \frac{3}{x^3}}{(2 + x^{-1/2})^3} = \lim_{x \rightarrow \infty} \frac{1}{2^3} = \frac{1}{8}$$

(b) $\lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{|x-1|}$

Solution:

For $x < 1$ we have $|x-1| = -(x-1)$ so $\lim_{x \rightarrow 1^-} \frac{(x-1)(x+1)}{|x-1|} = \lim_{x \rightarrow 1^-} -(x+1) = -2$

However for $x > 1$ we instead have $|x-1| = x-1$ so $\lim_{x \rightarrow 1^+} \frac{(x-1)(x+1)}{|x-1|} = \lim_{x \rightarrow 1^+} (x+1) = 2$.

The limits from left and right do not agree, so the limit DNE.

(c) $\lim_{x \rightarrow 0^-} e^{\frac{1}{x}}$

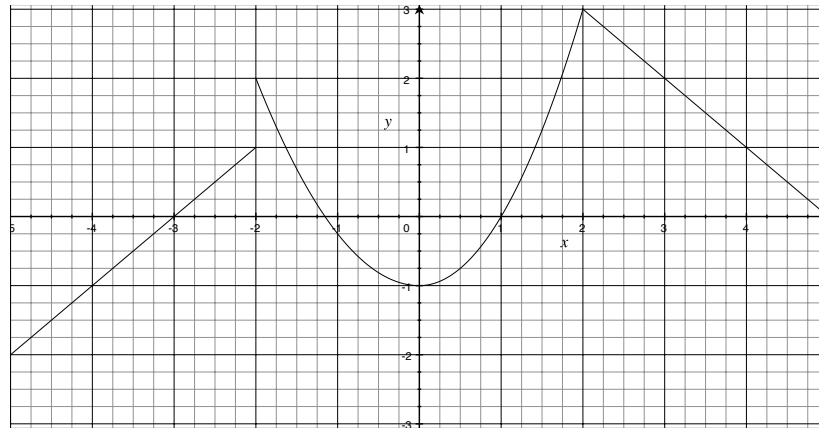
Solution:

Now $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$. Thus since e^x is continuous

$$\lim_{x \rightarrow 0^-} e^{\frac{1}{x}} = e^{\lim_{x \rightarrow 0^-} \frac{1}{x}} = \lim_{x \rightarrow -\infty} e^x = 0.$$

Name:

Problem A6: [6 pts] Consider the following graph of the function $f(x)$.



Find the following. If it doesn't exist say DNE. If it is infinite say whether its $+\infty$ or $-\infty$. (For this question, you do not need to justify your answer)

(a) $\lim_{x \rightarrow -2^-} f(x)$

Solution:

1

(b) $\lim_{x \rightarrow 0^+} f(x)$

Solution:

-1

(c) $\lim_{x \rightarrow -2} f(x)$

Solution:

DNE

(d) $f'(3)$

Solution:

-1

(e) At what points does $f(x)$ fail to be continuous?

Solution:

$x = -2$

(f) At what points does $f(x)$ fail to be differentiable?

Solution:

$x = -2, 2$

Name:

Problem A7: [5 pts] Find the tangent line to $y = \sqrt{x^2 + 3x}$ at the point (1,2). (If you compute a derivative, you must use the limit definition)

Solution:

The slope of the tangent line at (1,2) is

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\sqrt{(1+h)^2 + 3(1+h)} - 2}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{4+5h+h^2} - 2}{h} \frac{\sqrt{4+5h+h^2} + 2}{\sqrt{4+5h+h^2} + 2} \\ &= \lim_{h \rightarrow 0} \frac{5h + h^2}{h(2 + \sqrt{4+5h+h^2})} = \lim_{h \rightarrow 0} \frac{5+h}{2 + \sqrt{4+5h+h^2}} \\ &= \frac{5}{4}\end{aligned}$$

Therefore the tangent line is

$$y - 2 = \frac{5}{4}(x - 1)$$

or

$$y = \frac{5}{4}x + \frac{3}{4}$$

Name:

Problem A8: [9 pts] Use the power-rule to find the derivatives of the following functions

(a) $f(x) = 3x^4 + 2x^3 - 1$

Solution:

$$f'(x) = 12x^3 + 6x^2$$

(b) $g(t) = \sqrt{t} - \frac{2}{t^2}$

Solution:

$$g'(t) = \frac{1}{2\sqrt{t}} - \frac{4}{t^3}$$

(c) $h(x) = x^2 \left(x + \frac{1}{x}\right)$

Solution:

$$h(x) = x^3 + x \text{ so } h'(x) = 3x^2 + 1.$$