1. (20 points) Consider the following sets: \( A = \{1, 4, 9, 16\} \), \( B = \{-2, -1, 0, 1, 2\} \), \( C = \{1, 1, 2, 2, 2, 4\} \).

Answer:

a) Compute \( A - C \).
\( A - C = \{4, 9, 16\} \)

b) Compute \( (A \cup C) \cap B \).
\( A \cup C = \{1, 2, 4, 9, 16\} \) \( (A \cup C) \cap B = \{1, 2\} \)

c) Compute \( |C| \).
\( |C| = 3 \)

d) Compute the power set \( P(C) \).
\( P(C) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \)

2. (10 points) Is \( (p \rightarrow r) \land (q \rightarrow r) \) logically equivalent to \( (p \land q) \rightarrow r \)\

Answer:

No. One way to show this is to draw a truth table. But all we need to do is find any truth values of \( p \), \( q \), and \( r \) that give the two propositions different truth values. So let \( p = T \), \( q = F \), and \( r = F \). Then \( (p \land q) \rightarrow r \) evaluates to \( F \rightarrow F \), which is true. But \( (p \rightarrow r) \land (q \rightarrow r) \) evaluates to \( F \land T \), which is false.

3. (20 points) Explain whether the following arguments are valid.

a) Every positive real number is the square of a real number.
\(-1 \) is not a positive real number.
Therefore, \(-1 \) is not the square of a real number.
This is not a valid argument, even though all of the statements involved are true. It takes the form of the fallacy of denying the hypothesis.

b) If an object is either a plant or an animal, then it is a living thing.
A rock is not a living thing.
Therefore, a rock is not a plant.

This is valid. It takes the form $\forall x (P(x) \lor A(x)) \rightarrow L(x) \therefore \neg A(r)$ where $r$ is a rock, the domain of discourse is all objects, and $P(x)$, $A(x)$, and $L(x)$ mean that $x$ is a plant, an animal, and a living thing, respectively. The argument essentially uses Modus Tollens to conclude $\neg (P(r) \lor A(r))$, which is logically equivalent to $\neg P(r) \land \neg A(r)$. So by simplification, $\neg P(r)$. (Technically there is also a universal instantiation that occurs, but you’ll get credit even if you don’t explicitly say this.)

Answer:

4. (40 points) Consider the following quantified statements, for which the domain of discourse is $\mathbb{R}$. State whether each is true or false. Explain your answer in 1-2 sentences (you do not need to write a formal proof).

Answer:

a) $\exists x \forall y, xy = 0$
True. $x = 0$ works. For any $y \in \mathbb{R}$, it is true that $0y = 0$.

b) $\forall x \exists y, x + y = y$
False. If this statement is true, then $x = 0$. So it is false for $x = 1$.

c) $\forall x \forall y, x + y = y + x$
True. Addition of real numbers is commutative.

d) $\exists x \exists y, (x + 2y = 1) \land (2x + 4y = 3)$
False. If $x + 2y = 1$, then $2x + 4y = 2$. So both statements cannot be simultaneously true.

5. (40 points) Let $n \in \mathbb{Z}$. For this problem, we say that:
n is type 0 if there exists an integer k such that \( n = 3k \);

n is type 1 if there exists an integer k such that \( n = 3k + 1 \), and

n is type 2 if there exists an integer k such that \( n = 3k + 2 \).

**Answer:**

Every integer is either type 0, type 1, or type 2 (you may assume this fact without proof).

Prove the following statements, for which the domain of discourse is \( \mathbb{Z} \).

a) If \( x \) is type 1 and \( y \) is type 2, then \( x + y \) is type 0.

Let \( x = 3a + 1 \) and \( y = 3b + 2 \). Then \( x + y = 3a + 1 + 3b + 2 = 3(a + b + 1) \). So \( x + y \) is type 0.

b) If \( x \) is type 1 and \( y \) is type 2, then \( xy \) is type 2.

Let \( x = 3a + 1 \) and \( y = 3b + 2 \). Then \( xy = (3a + 1)(3b + 2) = 9ab + 3a + 3b + 2 = 3(3ab + a + b) + 2 \). So \( xy \) is type 2.

c) No integer is both type 1 and type 2.

Assume that \( x \) is both type 1 and type 2. Then \( x = 3a + 1 \) and \( x = 3b + 2 \) for some integers \( a, b \). So \( 3a + 1 = 3b + 2 \). So \( 3(a - b) = 1 \), and \( a - b = -\frac{1}{3} \). But this is a contradiction, because \( a - b \) is an integer. So no integer can be both type 1 and type 2.

d) If \( n = x^2 \), then \( n \) is either type 0 or type 1. Every integer is type 0, 1, or 2, so we can divide this into cases:

If \( x \) is type 0, then \( x = 3k \), and \( x^2 = 9k^2 = 3(3k^2) \) and \( x^2 \) is type 0.

If \( x \) is type 1, then \( x = 3k + 1 \) and \( x^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 \), and \( x^2 \) is type 1.

If \( x \) is type 2, then \( x = 3k + 2 \) and \( x^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1 \). So \( x^2 \) is type 1.

6. (30 points) Consider the functions below. For each one, determine if it is injective and if it is surjective. Prove your assertions.

**Answer:**

a) \( f : \mathbb{Z} \to \mathbb{Z} \) given by \( f(x) = 2x + 3 \)
This is injective. If \(2x + 3 = 2y + 3\), then \(2x = 2y\), and so \(x = y\). It is not surjective, because if \(2x + 3 = 0\), then \(x = \frac{-3}{2}\), which is not an integer.

b) \(g : \mathbb{R} \rightarrow \mathbb{R}\) given by \(g(x) = 2x + 3\)

This is injective for the same reason as in part a. It is also surjective. If \(y \in \mathbb{R}\), then \(2x + 3 = y\) has a solution: \(x = \frac{y-3}{2}\). So \(g\left(\frac{y-3}{2}\right) = y\).

c) \(h : \mathbb{R} \rightarrow \mathbb{Z}\) given by \(h(x) = \lfloor x \rfloor\)

This is not injective, as \(\lfloor 0 \rfloor = 0\) and \(\lfloor 0.5 \rfloor = 0\). It is surjective. For any \(y \in \mathbb{Z}\), \(\lfloor y \rfloor = y\).