1. (16 points) Page 1, Problem 7.

(a) Circle whether each of the following integrals is proper or improper. Evaluate each of the following integrals if they exist. If they do not exist, then say so. Show all your work.

(i) Circle one: proper or improper

$$\int_0^\infty 4x^3 e^{-x^4}\,dx$$

Solution: Since the upper limit is ∞ this is an **improper** integral.

$$\int_0^\infty 4x^3 e^{-x^4}\,dx = \lim_{a\to\infty}\int_0^a 4x^3 e^{-x^4}\,dx$$

Let $u = -x^4$ then $du = -4x^3$. If $x = 0$ then $u = -0^4 = 0$ and if $x = \infty$ then $u = -(\infty)^4 = -\infty$ so with a u-substitution we get

$$= \lim_{a\to-\infty}\int_0^a -e^u du = \lim_{a\to-\infty} -e^u\Big|_0^a = \lim_{a\to-\infty} -e^a - -e^0 = 0 + e^0 = 1.$$

(ii) Circle one: proper or improper

$$\int_4^7 \frac{1}{(x-5)^3}dx =$$

Solution: Since $\frac{1}{(x-5)^3}$ has a vertical asymptote at $x = 5$ (where the denominator is zero) the integral is **improper** (since 5 is inside the limits of integration.)

So $\int_4^7 \frac{1}{(x-5)^3}dx = \lim_{a\to5^-}\int_4^a \frac{1}{(x-5)^3}dx + \lim_{b\to5^+}\int_b^7 \frac{1}{(x-5)^3}dx =$

The integral converges only if both of the integrals above converge. We'll try the left integral first.

$$\lim_{a\to5^-}\int_4^a \frac{1}{(x-5)^3}dx = \lim_{a\to5^-}\int_4^a (x-5)^{-3}\,dx = \lim_{a\to5^-}\frac{(x-5)^{-2}}{-2}\Bigg|_4^a = \lim_{a\to5^-}\frac{1}{(-2)(a-5)^2} - \frac{1}{(-2)(4-5)^2}$$
$$= -\infty + \frac{1}{2} = -\infty$$

Since this integral diverges, $\int_4^7 \frac{1}{(x-5)^3}dx$ must also diverge.

(b) For which values of p does $\int_1^\infty \frac{1}{x^p}x$ converge, and for which values of p does it diverge? You can answer this from memory.

Solution: The integral converges for $p > 1$ and diverges for $p \le 1$

2. (10 points) Page 2, Problem 1.
Compute the length of the curve $y = \frac{x^2}{8} - \ln x$ from $x = 1$ to $x = 2$
Solution:
To compute arc length with cartesian coordinates you must integrate $\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$. So the lets find out what that is.

$$\frac{dy}{dx} = \frac{x}{4} - \frac{1}{x}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{x}{4} - \frac{1}{x}\right)^2 = \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \left(\frac{x}{4} - \frac{1}{x}\right)^2 + 1 = \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2} + 1 = \left(\frac{x}{4} + \frac{1}{x}\right)^2$$

Using the above information we get that

$$L = \int_1^2 \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}dx = \int_1^2 \sqrt{\left(\frac{x}{4} + \frac{1}{x}\right)^2}dx = \int_1^2 \frac{x}{4} + \frac{1}{x}dx = \frac{x^2}{8} + \ln|x|\Big|_1^2$$
$$= \left(\frac{2^2}{8} + \ln 2\right) - \left(\frac{1^2}{8} + \ln 1\right) = \left(\frac{1}{2} + \ln 2\right) - \left(\frac{1}{8} + 0\right) = \frac{3}{8} + \ln 2$$

3. (10 points) Page 3, Problem 2. Compute the area of the surface obtained by rotating the curve $y = x^3$ from $x = 0$ to $x = 1$ about the x-axis.

Solution:

By graphing the curve one can easily see that the radius caused by revolving the curve about the x-axis is the y-value. To integrate $\int 2\pi r ds$ (recall that $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ or dy) we get the integral

$$\int 2\pi r ds = \int_0^1 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_0^1 x^3 \sqrt{1 + (3x^2)^2} dx = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} dx$$

So we must do a u-substitution:

$u = 1 + 9x^4$ so $du = 36x^3 dx \Rightarrow \frac{1}{36} du = x^3 dx$

Also, $x = 0 \Rightarrow u = 1 + 9 \cdot 0^4 = 1$ and $x = 1 \Rightarrow u = 1 + 9 \cdot 1^4 = 10$

So we get

$$\frac{2}{36}\pi \int_1^{10} \sqrt{u} du = \frac{\pi}{18} \int_1^{10} u^{\frac{1}{2}} du = \frac{\pi}{18} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} \Big|_1^{10} = \frac{\pi}{27} \left(10^{3/2} - 1^{3/2}\right)$$

4. (10 points) Page 4, problem 3.

Compute the area of the surface obtained by rotating the curve $y = x^2$ from $x = 0$ to $x = 1$ about the $y - axis$.

Solution:

By looking at a picture one can easily see for the this curve that the radius will be the x value, so we get

$$SA = \int 2\pi r ds = 2\pi \int_0^1 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_0^1 x \sqrt{1 + 4x^2} dx$$

So let $u = 1 + 4x^2$ and then $du = 8xdx \Rightarrow \frac{1}{8} du = xdx$

Also $x = 0 \Rightarrow u = 1 + 4 \cdot 0^2 = 1$ and $x = 1 \Rightarrow u = 1 + 4 \cdot 1^2 = 5$

Thus

$$SA = \frac{2\pi}{8} \int_1^5 \sqrt{u} du = \frac{2\pi}{8} \cdot \frac{2}{3} u^{3/2}\Big|_1^5 = \frac{\pi}{12} \left(5^{3/2} - 1^{3/2}\right)$$

5. (15 points) Page 5, problem 7.

(a) Find the area under the curve represented by the parametric equations

$$x = 2\cos t \qquad y = 5\sin t$$

From $t = \pi$ to $t = 0$. Use $\sin^2 x = \frac{1-\cos(2x)}{2}$ or $\cos^2 x = \frac{1+\cos(2x)}{2}$ if necessary.

Solution: To find the area beneath a parametric curve we must integrate $\int y \cdot x' dt$ So we get

$$\int_{\pi}^{0} (5\sin t)(-2\sin t)\,dt = -10\int_{\pi}^{0} \sin^2 t dt = -10\int_{\pi}^{0} \frac{1-\cos(2t)}{2} dt = -5\int_{\pi}^{0} 1-\cos(2t)\,dt$$

$$= -5\left(t - \frac{1}{2}\sin(2t)\right)\Big|_{\pi}^{0} = -5\left[\left(0 - \frac{1}{2}\sin(0)\right) - \left(\pi - \frac{1}{2}\sin(2\pi)\right)\right]$$
$$= -5\left[(0-0) - (\pi - 0)\right] = -5(-\pi) = 5\pi$$

(b) Find the length of the curve, Cm from $t = 0$ to $t = \pi$ where C is given by the parametric equations

$$x = 2\cos t \qquad y = 2\sin t$$

Solution: To find arclength in parametric coordinates we must integrate $\sqrt{(x'(t))^2 + (y'(t))^2}$ So we get

$$\int_0^{\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2} dt = \int_0^{\pi} \sqrt{4\sin^2 t + 4\cos^2 t} dt = \int_0^{\pi} \sqrt{4} dt = (2t)|_0^{\pi} = 2\pi$$

6. (10 points) Page 6, problem 1.

(a) Represent the point $(r, \theta) = \left(-2, \frac{\pi}{2}\right)$ in Cartesian Coordinates.

Solution: $x = r\cos\theta = -2\cos\left(\frac{\pi}{2}\right) = -2 \cdot 0 = 0$

$y = r\sin\theta = -2\sin\left(\frac{\pi}{2}\right) = - - 2 \cdot 1 = -2$

So the point is $(0, -2)$.

(b) Represent the point $(x, y) = (-3, 3)$ in polar coordinates.

Solution:

$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 3^2} = \sqrt{18}$

$\theta = \arctan\left(\frac{3}{-3}\right) = \arctan(-1) = -\frac{\pi}{4}$

Notice that the polar point $\left(\sqrt{18}, -\frac{\pi}{4}\right)$ lies in the 4th quadrant. But $(-3, 3)$ is in the 2nd quadrant. So must adjust either r or θ to get the point.

If we change r we will get $r = -\sqrt{18}$ (instead of $\sqrt{18}$). So the point is: $\left(-\sqrt{18}, -\frac{\pi}{4}\right)$

If we change θ we must add π. So we get $\theta = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$. So the point is:$\left(\sqrt{18}, \frac{3\pi}{4}\right)$

7. (20 points) Page 7, problem 2. Given the polar curves $r = 4\sin\theta$ and $r = 2\sqrt{3}$, find the following.

(a) The cartesian equations of the two curves.

Solution:

The Curve $r = 4\sin\theta$ is a circle that opens upwards on the y-axis with a diameter of 4. So it has a radius of 2, and its center is $(0, 2)$. So its cartesian equation is $x^2 + (y-2)^2 = 2^2$.

The curve $r = 2\sqrt{3}$ is a circle with radius $2\sqrt{3}$ centered at the origin. So its cartesian equation is $x^2 + y^2 = \left(2\sqrt{3}\right)^2 = 12$.

Note. You can solve both of these for y, but thats unnecessary.

(b) All intersection points of the two curves (It helps to sketch the curves)

Solution: Need to find when $4\sin theta = 2\sqrt{3} \Rightarrow \sin\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$ so

$$\theta = \arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3}$$

$\frac{\pi}{3}$ is a first quadrant angle. There is a second angle where $\sin\theta = \frac{\sqrt{3}}{2}$ its the same angle in the second quadrant. So the second angle is

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Page 8, problem 2c.

(c) The area of the region that lies inside the first curve and outside the second. Use $\sin^2 x = \frac{1-\cos(2x)}{2}$ or $\cos^2 x = \frac{1+\cos(2x)}{2}$ if necessary.

Solution: Recall that to find the area inside a polar curve one must integrate $\frac{1}{2}r^2$. Also recall that we calculated the limits of integration in part (b). So we get

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (4\sin\theta)^2 - \left(3\sqrt{2}\right)^2 d\theta = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 16\sin^2\theta - 12d\theta = 2\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 4\sin^2\theta - 3d\theta$$

$$= 2\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 2\left(1-\cos(2\theta)\right) - 3d\theta = 2\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} -1 - 2\cos(2\theta)\, d\theta = -2\left[\theta - \sin(2\theta)\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$= -2\left[\left(\frac{2\pi}{3} + \sin\left(\frac{4\pi}{3}\right)\right) - \left(\frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right)\right)\right] = -2\left[\left(\frac{2\pi}{3} + -\frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\right]$$

$$= -2\left(\frac{\pi}{3} - \sqrt{3}\right)$$

8. (10 points) Page 9, problem 2. For each sequence, find a forula for the general term a_n. For example, answer n^2, if given the sequence: 1, 4, 9, 16, ...

(a)

$$\frac{1}{3}, \frac{2}{9}, \frac{4}{81}, \ldots$$

Solution: First look at the numerator. It is obvious that the number for $a_n = n$. Now look at the denominator. For each succesive term we simply multiply by 3 which gives us that

$$a_n = \frac{n}{3^n}$$

(b)

$$1, -2, +3, -4, \ldots$$

Solution: If we look at the absolute value of each term, it increases by 1 each time. So $|a_n| = n$. The sequence also alternates so we need to multiply by (-1) each time. So our choices are $a_n = (-1)^n \cdot n$ or $a_n = (-1)^{n+1} \cdot n$

Since the first term is positive we need the exponent of -1 to be even for $n = 1$ which gives us that

$$a_n = (-1)^{n+1} \cdot n$$

9. (10 points) Determine whether each of the following sequences converges or diverges. If a sequence converges compute its limit. If a sequence diverges state whether it diverges to $+\infty$, $-\infty$ or neither.

(a) $\left\{\dfrac{3+5n^2}{n+n^2}\right\}$

Solution: The sequence has an associated function $f(x) = \dfrac{3+5x^2}{x+x^2}$

Since $\lim_{x\to\infty} f(x) = 5$ we get that the sequence also converges to 5.

(b) $\left\{\dfrac{2^n}{n!}\right\}$

Solution: Look at the sequence.

$a_1 = \frac{2}{1}$

$a_2 = \frac{2\cdot 2}{2\cdot 1}$

$a_3 = \frac{2\cdot 2\cdot 2}{3\cdot 2\cdot 1} = \left(\frac{2}{3}\right)\frac{2\cdot 2}{2\cdot 1} = (\text{ something less than 1 })\, a_2 < a_2$

$a_4 = \frac{2}{4}a_3 < a_3$

$\vdots$

$a_n = \frac{2}{n}a_{n-1} < n-1$

So after the 2nd term each element of the sequence is smaller than the one before, and they are all positive. So it converges to 0.

(c) $\left\{\left(1+\frac{2}{n}\right)^n\right\}$

Solution:The sequence has the associated function $f(x) = \left(1+\frac{2}{x}\right)^x$.

Recall that if $\lim_{x\to\infty} \ln(f(x)) = L$ then $\lim_{x\to\infty} f(x) = e^L$.

So we consider $\lim_{x\to\infty} \ln\left(1+\dfrac{2}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\dfrac{2}{x}\right)$. This limit is an indeterminate form $(\infty\cdot 0)$. So we use L'Hopital's rule to get

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\cdot\frac{-2}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}} = 2$$

So we get that the sequence converges to e^2.