Solutions Math 142 midterm 1 Spring 2004

1. (20 pts) Consider the function $f(x) = x^3 + 3x^2 - 2$.

   (a) On which interval(s) is f increasing?

   $$f'(x) = 3x^2 + 6x = 3x(x+2)$$

   f' is positive on $(-\infty, -2)$ and $(0, \infty)$, so f is increasing on $(-\infty, -2)$ and $(0, \infty)$.

   (b) On which interval(s) is f decreasing?

   f' is negative on $(-2, 0)$ and this is where f is decreasing.

   (c) On which interval(s) is f concave up?

   $$f''(x) = 6x + 6 = 6(x+1)$$

   f'' is positive on $(-1, \infty)$ so this is where f is concave up.

   (d) On which interval(s) is f concave down?

   f'' is negative on $(-\infty, -1)$ and this is where f is convave down.

   (e) For which value(s) of x does f have a local maximum?

   The critical points for f are at $x = -2$ and $x = 0$. At $x = -2$, f'' is negative so by the second derivative test, f has a maximum at $x = -2$.

   (f) For which value(s) of x does f have a local minimum?

   At $x = 0$, f'' is positive so by the second derivative test, f has a local minimum at $x = 0$

   (g) Using the information above, sketch the graph of

   $$y = x^3 + 3x^2 - 2$$

   on the axes below.

2. (19 points) Determine the dimensions and the area of the largest rectangle that can be inscribed inside the ellipse defined by

$$\frac{x^2}{4} + y^2 = 2$$

Assume the rectangle inside the ellipse has top right corner at a point (x, y) on the ellipse. Then the area of the rectangle is

$$A = 4xy$$

Since (x, y) is on the ellipse and y is positive we have that $y = \frac{\sqrt{8-x^2}}{2}$ so

$$A = 4x\frac{\sqrt{8-x^2}}{2} = 2x\sqrt{8-x^2}$$

A is defined for $x \in [0, \sqrt{8}]$. A obtains its maximum at either a critical point or at the endpoints of this interval.

$$\begin{aligned} A' &= 2\sqrt{8-x^2} + 2x(8-x^2)^{-1/2}\frac{1}{2}(-2x) \\ &= 2\sqrt{8-x^2} - 2x^2\frac{1}{\sqrt{8-x^2}} \\ &= \frac{2}{\sqrt{8-x^2}}((8-x^2)-x^2) \\ &= \frac{4}{\sqrt{8-x^2}}(4-x^2) \end{aligned}$$

Therefore $A' = 0$ when $x = 2$ so this is a critical point. At this point $y = 1$ and there area A is therefore $A = 8$. The dimensions of the box are 4 by 2.

3. (5 points) Using Newton's method to approximate the root of $x^4 + x - 4$ on the interval $[1, 2]$ assume that $x_1 = 1$. Find x_2.

By Newton's method we let

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In this case $f(x) = x^4 + x - 4$ so $f(1) = -2$ and $f'(x) = 4x^3 + 1$ so $f'(1) = 5$ Therefore

$$x_2 = 1 - \frac{-2}{5} = \frac{7}{5}$$

4. Using x_1 as the initial approximation, mark on this graph the next approximation x_2 to the solution of

$$f(x) = 0$$

5. Find the most general antiderivatives for the following:

(a) $f(x) = \frac{1}{x^2} + x^2 + \frac{1}{2x}$
$\int f(x)dx = -x^{-1} + \frac{1}{3}x^3 + \frac{1}{2}\ln x + C$

(b) $f(x) = \frac{8}{1+x^2}$
$\int f(x)dx = 8\arctan x + C$

(c) $f(x) = 5\sqrt{x^3} + e^2$
$\int f(x)dx = 2x^{\frac{5}{2}} + e^2 x + C$

(d) $f(x) = \sec(x)\tan(x) + \sin(3x)$
$\int f(x)dx = \sec(x) - \frac{1}{3}\cos(3x) + C$

6. (15 points) Evaluate the Riemann sum R_4 for

$$\int_1^3 (4 + \frac{1}{x})dx$$

using $n = 4$ subintervals of equal length, and using right endpoints. You do not have to simplify your answer.

$$\begin{aligned} R_4 &= \Delta x(f(3/2) + f(2) + f(5/2) + f(3)) \\ &= \frac{1}{2}(4 + \frac{2}{3} + 4 + \frac{1}{2} + 4 + \frac{2}{5} + 4 + \frac{1}{3}) \\ &= 8 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \end{aligned}$$

The correct statement is

$$R_4 \leq \int_1^3 (4 + \frac{1}{x})dx$$

7. Use part 1 of the Fundamental Theorem of Calculus to find the derivative of the functions:

   (a) $g(x) = \int_2^x \sqrt{1+t^2}dt$

   $$g'(x) = \sqrt{1+x^2}$$

   (b) $g(x) = \int_x^{x^2} e^t dt$
   First break $g(x)$ into two pieces:

   $$g(x) = \int_x^0 e^t dt + \int_0^{x^2} e^t dt = -\int_0^x e^t dt + \int_0^{x^2} e^t dt$$

   Then $g'(x) = -e^x + e^{x^2}(2x)$ by the chain rule.

8. (10 points) Evaluate the following definite integrals:

   (a) $\int_0^1 \frac{(1+3x-x^2)}{\sqrt{x}}dx$

   $$\begin{aligned} &= \int_0^1 x^{-\frac{1}{2}} + 3x^{\frac{1}{2}} - x^{\frac{3}{2}}dx \\ &= 2x^{\frac{1}{2}} + 2x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}]_0^1 \\ &= 2+2-\frac{2}{5} = 3.6 \end{aligned}$$

   (b) $\int_{\ln(3)}^{\ln(6)} 2e^x dx$

   $$= 2e^x]_{\ln(3)}^{\ln(6)} = 2(6-3) = 6$$