MATH 142

MIDTERM EXAM I SOLUTIONS

October 2, 2003

1. (20 pts) Consider the function $f(x) = -x^3 + 3x^2 - 1$.

(a) On which interval(s) is f increasing?

$$f'(x) = -3x^2 + 6x = -3x(x-2).$$

Therefore $f'(x) = 0$ when $x = 0$ or $x = 2$. Thus there are three intervals to check: $(-\infty, 0), (0, 2)$ and $(2, \infty)$. f is increasing when $f'(x) > 0$. Checking test numbers, we see that $f'(x) > 0$ when

$$x \in (0, 2).$$

(b) On which interval(s) is f decreasing?

Using the work above, and checking test numbers, we see that f is decreasing (i.e. $f'(x) < 0$) when

$$x < 0 \quad \text{or } x > 2.$$

(c) On which interval(s) is f concave up?

$$f''(x) = -6x + 6 = -6(x-1).$$

Therefore $f''(x) = 0$ when $x = 1$. Thus there are two intervals to check for concavity: $(-\infty, 1)$ and $(1, \infty)$. Checking test numbers, f is concave up (i.e. $f''(x) > 0$) when

$$x < 1.$$

(d) On which interval(s) is f concave down? Using the work above, f is concave down (i.e. $f''(x) < 0$) when

$$x > 1.$$

Note that because concavity changes at 1, $x = 1$ is an inflection point.

(e) For which value(s) of x does f have a local maximum?

There are two critical numbers from part (a): $x = 0$ and $x = 2$. $f''(2) = -6 < 0$, so by the second derivative test, $x = 2$ is a local maximum.

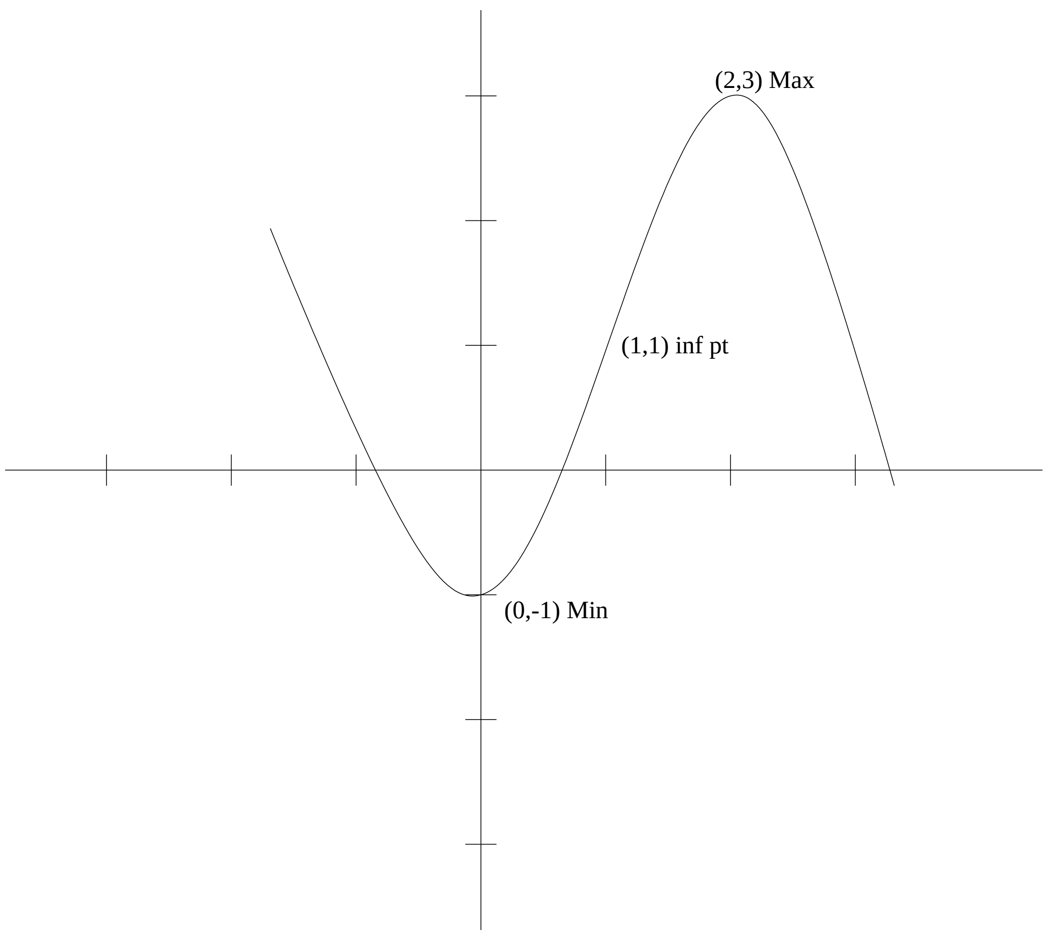
(f) For which value(s) of x does f have a local minimum?

$x = 0$ is a local min by the second derivative test because $f''(0) = 6 > 0$.

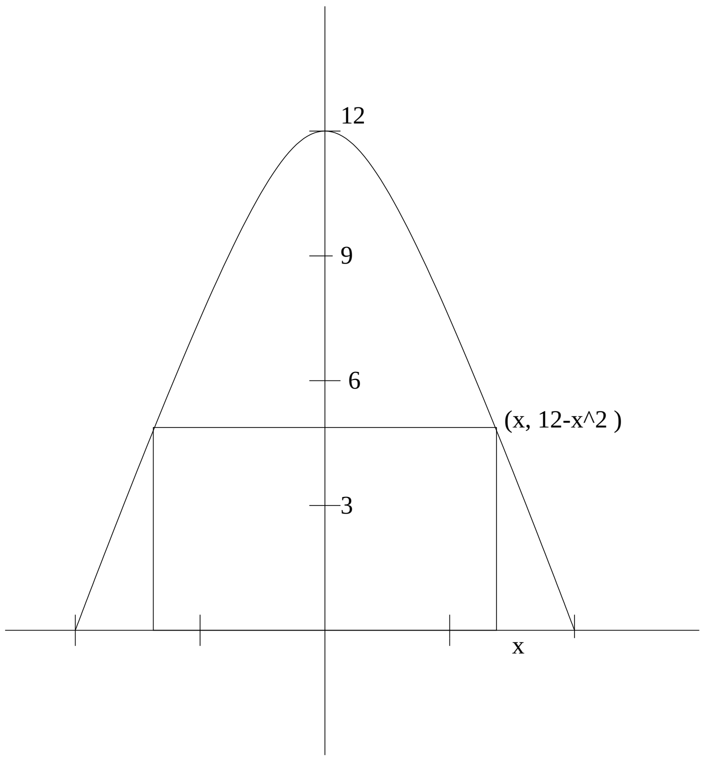
(g) Using the information above, sketch the graph of

$$y = -x^3 + 3x^2 - 1$$

on the axes below. Label the coords of all **max/mins** and **inflection points**. (On the axes below, 1 mark = 1 unit.)



2. (20 pts) Find the dimensions of the rectangle of largest area that has its base on the $x-$axis and its other two vertices above the $x-$axis and lying on the parabola $y = 12 - x^2$.



The base of the above rectangle has length $2x$, and the height is $12 - x^2$. Therefore the area is

$$A(x) = 2x(12 - x^2) = 24x - 2x^3.$$

To find the max, set

$$A'(x) = 24 - 6x^2 = 0$$

which gives $x^2 = 4$, so $x = \pm 2$. As drawn, we are taking $x > 0$, so the only relevant value is $x = 2$. This is indeed a maximum because $A''(2) = -24 < 0$.
Thus the dimensions are width $= 2(2) = 4$ and height $= 12 - (2)^2 = 8$.

3. (15 pts) Find all asymptotes (horizontal and vertical) for the graph of the function

$$f(x) = \frac{x^3+1}{x^2(x-1)}.$$

Horizontal Asymptotes:

$$\lim_{x\to\infty} \frac{x^3+1}{x^3-x^2} = \lim_{x\to\infty} \frac{x^3}{x^3} = 1.$$

Therefore $y = 1$ is a horizontal asymptote. Simlarly,

$$\lim_{x\to-\infty} f(x) = 1,$$

so this gives the same asymptote.

Vertical Asymptotes:

Note $x = 0$ and $x = 1$ make the denominator 0, so these are the potential vertical asymptotes.

$$\lim_{x\to 0^+} \frac{x^3+1}{x^2(x-1)} = \frac{1}{(\text{tiny pos \#})(-1)} = -\infty.$$

Because this limit is infinite, $x = 0$ is indeed a vertical asymptote. Similarly,

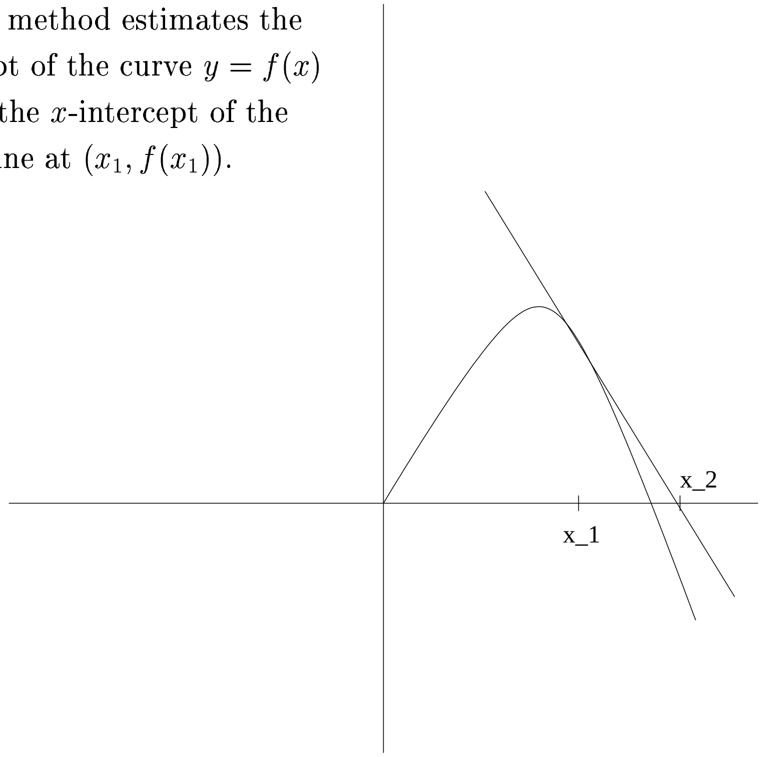
$$\lim_{x\to 1^+} f(x) = \frac{2}{(\text{tiny pos \#})} = \infty,$$

so $x = 1$ is also a vertical asymptote.

4. (14 pts)

(a) On the graph below, sketch the approximation x_2 given by Newton's method to the solution of $f(x) = 0$, if x_1 is the initial approximation.

Newton's method estimates the x-intercept of the curve $y = f(x)$ by using the x-intercept of the tangent line at $(x_1, f(x_1))$.



(b) Using Newton's method to approximate $\sqrt{3}$ (i.e. to solve $x^2 - 3 = 0$), suppose $x_1 = 1$. Find x_2.

Let $f(x) = x^2 - 3$, so $f'(x) = 2x$. Then

$$\sqrt{3} \approx x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{-2}{2} = 2.$$

5. (16 pts) Find the most general antiderivatives for the following:

(a) $f(x) = \dfrac{x}{5} + x^3$

ANSWER: $\frac{x^2}{10} + \frac{x^4}{4} + C$

(b) $f(x) = \dfrac{13}{\sqrt{1-x^2}}$

ANSWER: $13 \arcsin(x) + C$.

(c) $f(x) = \cos(2x) + e^{-x}$

ANSWER: $\frac{1}{2}\sin(2x) - e^{-x} + C$.

(d) $f(x) = \sec^2(x)$

ANSWER: $\tan(x) + C$.

6. (15 pts) Evaluate the Riemann sum R_8 for

$$\int_0^2 (4 - x^2)dx$$

using $n = 8$ subintervals of equal length, and using right endpoints. You do not have to simplify your answer.

$$R_8 = \frac{1}{4}[(4-(1/4)^2) + (4-(2/4)^2) + (4-(3/4)^2) + (4-(4/4)^2)$$
$$+(4-(5/4)^2) + (4-(6/4)^2) + (4-(7/4)^2) + (4-(8/4)^2)].$$

Circle the correct statement:

$$R_8 \le \int_0^2 (4-x^2)dx \qquad\qquad R_8 \ge \int_0^2 (4-x^2)dx$$

The function $4-x^2$ is decreasing on $[0, 2]$, so if you draw a picture you will see that rectangles with right endpoints are underneath the curve, so the actual area is larger than the rectangle approximation. (I.e. the first inequality is the correct one).